Twistor inspired developments in perturbative QCD

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29 November 2006
Hard processes in Hadron-Hadron collisions

\[ \sigma(Q^2) = \sum_{i,j} \left[ \hat{\sigma}_{ij} \left( \alpha_s(\mu^2), \frac{\mu_R^2}{Q^2}, \frac{\mu_F^2}{Q^2} \right) \otimes f_i^p(\mu_F^2) \otimes f_j^p(\mu_F^2) \right] \]

- partonic cross sections \( \hat{\sigma}_{ij} \)
- parton distributions \( f_i \)
- renormalization/factorization scale \( \mu_R/\mu_F \)
- + parton shower + hadronisation model
The renormalisation scale $\mu_R$ is introduced when redefining the bare fields in terms of the physical fields at scale $\mu_R$. It is unphysical - and the answer shouldn't depend on it - but does because we work at a fixed order in perturbation theory. Therefore, you can choose any value (within reason). Typical values are the hard scale in the process $\mu_R \sim E_T$.

Example: $pp \rightarrow \text{jet} + X$ at LO $\alpha_s^2$ for various values of $\mu_R$ compared to $\mu_R = E_T$.
The unphysical scales - $\mu_F$

The factorisation scale $\mu_F$ is introduced when absorbing the divergence from collinear radiation into the parton densities. It is unphysical - and the answer shouldn't depend on it - but does because we work at a fixed order in perturbation theory. **Typically** we think of radiation at a transverse energy $> \mu_F$ as being detectable so that $\mu_F \sim E_T$ is a reasonable choice.

![Graph showing ratios for various $\mu_F$ values compared to $\mu_F = E_T$.

Example: $pp \rightarrow \text{jet} + X$ at LO

The effective parton-parton luminosities for various values of $\mu_F$ compared to $\mu_F = E_T$ at $\eta_1 = \eta_2 = 0$
Unphysical scale dependence

- typically, NLO reduces scale uncertainty by factor 2 over LO
- √ maybe to ±30%
- typically, NNLO reduces scale uncertainty by factor 2 over NLO
- √ maybe ±10%
- ✗ won’t know till you do it

✓ plus many other improvements in modelling hard scattering at NNLO
## State of the Art

### Relative Order

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**LO** ✓ matrix elements automatically generated up to $2 \to 8$ or more

✓ plus automatic integration over phase space

HELAC/PHEGAS, MADGRAPH/MADEVENT, SHERPA/AMEGIC++, COMPHEP, GRACE, ...

✓ able to interface with parton showers - **CKKW**

✓ very good for estimating importance of various processes in different models - properly populate phase space with multiple hard objects

✗ rate very dependent on choice of renormalisation/factorisation scales
State of the Art

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NLO ✓  parton level integrators available for most $2 \to 2$ Standard Model and MSSM processes for some time

✓ extensively used at LEP, TEVATRON and HERA EVENT, JETRAD, MCFM, DISENT, etc

✓ reduced renormalisation scale uncertainty

✓ can be matched with parton shower MC@NLO – Frixione, Webber
State of the Art

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NLO ✓

some $2 \rightarrow 3$ processes available at NLO

- e.g. backgrounds \( pp \rightarrow 3 \text{ jets}, V + 2 \text{ jets}, \gamma\gamma + \text{ jet}, V + b\bar{b} \)
- as well as signals \( pp \rightarrow t\bar{t}H, b\bar{b}H, qqH, HHH, \bar{t}t \bar{t} \bar{j} \)

✗ many still missing \(VV + \text{ jet}, t\bar{t}+ \text{ jet}, \) etc

✗ understood how to do, but tedious and painstaking
# State of the Art

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**NLO**  
- $\times$ no 2 → 4 LHC cross sections known
- $\times$ need to extend range of available calculations to e.g. $pp \rightarrow W + \text{multijets}$ that are backgrounds to **New Physics**
  - $\checkmark$ 4 gluons@one-loop, Ellis, Sexton, 1986, $\sigma_{2j}$, 1992
  - $\checkmark$ 5 gluons@one-loop, Bern, Dixon, Kosower, 1993, $\sigma_{3j}$, 2000
  - $\checkmark$ 6 gluons@one-loop, many authors, 2006 $\sigma_{4j}$, 2000
- $\times$ need a more efficient way of evaluating loop contributions and constructing $\sigma$

Twistor inspired developments in perturbative QCD – p.9
How to calculate scattering amplitudes

1. Off-shell methods

   Traditional Feynman diagram approach

2. On-shell methods

   Based on S-matrix ideas of 1960’s but recently inspired by Witten’s proposal to relate perturbative gauge theory amplitudes to topological string theory in twistor space

   Witten, hep-th/0312171

   ⇒ new ways to calculate amplitudes in massless gauge theories:
Off-shell methods

Traditional Feynman diagram approach for off-shell Greens functions

- Direct link to Lagrangian
- Easy to adapt to any model
- Easy to include massive particles with/without spin
- Easy to automate
  - tree-level packages: Madgraph/Grace/CompHep/…
- Off-shell Berends-Giele recursion relations
  - tree-level packages: Alpgen/HELAC/PHEGAS/…
- Many Feynman diagrams
- Large cancellations between diagrams
- Loop amplitudes manpower intensive
Multi-jet production at the LHC using **HELAC/PHEGAS**

Draggiotis, Kleiss, Papadopoloulos

<table>
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<th># of jets</th>
<th>2</th>
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<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>14</td>
<td>28</td>
<td>36</td>
<td>64</td>
<td>78</td>
<td>130</td>
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<tr>
<td>total # of processes</td>
<td>126</td>
<td>206</td>
<td>621</td>
<td>861</td>
<td>1862</td>
<td>2326</td>
<td>4342</td>
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<tr>
<td>$\sigma (nb)$</td>
<td>-</td>
<td>91.41</td>
<td>6.54</td>
<td>0.458</td>
<td>0.030</td>
<td>0.0022</td>
<td>0.00021</td>
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<tr>
<td>% Gluonic</td>
<td>-</td>
<td>45.7</td>
<td>39.2</td>
<td>35.7</td>
<td>35.1</td>
<td>33.8</td>
<td>26.6</td>
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- The number of Feynman diagrams for an $n$ gluon process increases very quickly with $n$
- for the 10 gluon amplitude there are 10,525,900 diagrams
- Feynman diagrams very inefficient for many legs
- Control the quantum numbers of the scattering particles
On-shell methods

- New (and puzzling) insights into field theory amplitudes
  - new ways to calculate amplitudes in massless gauge theories:
    - MHV rules
      - NEW analytic results for some QCD tree amplitudes with any number of legs
    - BCF on-shell recursion relations
      - NEW compact results for some multileg QCD tree amplitudes
    - Unitarity and cut-constructibility
      - NEW analytic one-loop amplitudes in massless supersymmetric theories
    - Recursive derivation of rational terms
      - NEW analytic one-loop amplitudes for multigluon amplitudes

Cachazo, Svrcek and Witten
Britto, Cachazo and Feng (and Witten)
Bern, Dixon, Dunbar, Kosower; Britto, Cachazo and Feng; ...
Spinor Helicity Formalism

In Weyl (chiral) representation, each helicity state is represented by a bi-spinor \((a = 1, 2)\)

\[
\begin{align*}
 u_+(p) &= \lambda_{pa}, \\
 \overline{u_+(p)} &= \tilde{\lambda}_{\dot{p} \dot{a}}, \\
 u_-(p) &= \tilde{\lambda}_{\dot{p}}, \\
 \overline{u_-(p)} &= \lambda_p
\end{align*}
\]

so that

\[
\begin{align*}
 \langle ij \rangle &= \overline{u_-(p_i)u_+(p_j)} = \lambda_i^a \lambda_j^a = \epsilon_{ab} \lambda_i^a \lambda_j^b \\
 [ij] &= \overline{u_+(p_i)u_-(p_j)} = \tilde{\lambda}_{i \dot{a}} \tilde{\lambda}_{j \dot{a}} = -\epsilon_{\dot{a} \dot{b}} \tilde{\lambda}_{i \dot{a}} \tilde{\lambda}_{j \dot{b}}
\end{align*}
\]

We can write massless vector

\[
p_{a \dot{a}} \equiv p_\mu \sigma_\mu^a_{a \dot{a}} = \lambda_{pa} \tilde{\lambda}_{p \dot{a}}
\]
Spinor Helicity Formalism

- Polarisation vectors for particle \( i \):

\[
\varepsilon_{i a \bar{a}} = \frac{\lambda_{i a} \tilde{\eta}_{\bar{a}}}{[\tilde{\lambda}_{i} \tilde{\eta}]}, \quad \varepsilon_{i a \bar{a}}^+ = \frac{\eta_{a} \tilde{\lambda}_{i \bar{a}}}{\langle \eta \lambda_{i} \rangle}
\]

- For real momenta in Minkowski space,

\[
\tilde{\lambda} = \lambda^*
\]

\[
\langle ij \rangle^* = -[ij]
\]

- For space-time signature \((+,-,-,-)\), \( \tilde{\lambda}, \lambda \) are real and independent

- Amplitudes are functions of the \( \lambda_{i} \) and \( \tilde{\lambda}_{i} \)
Gluonic helicity amplitudes

- A gluon has either positive or negative helicity (right-handed or left-handed)

- A multigluon amplitude can be characterised by the helicity of the gluons

- There will $n_+$ positive helicities and $n_-$ negative helicities.

- The order of helicities matters:
  $- - + + + + + + +\ldots$ is not the same as $- + - + + + + + +\ldots$ etc.
Gluonic helicity amplitudes

Each row describes scattering with $n_+$ positive helicities and $n_-$ negative helicities.
Each circle represents an allowed helicity configuration - from all positive on the right to all negative on the left.
Gluonic helicity amplitudes

For example, the result of computing the 25 diagrams for the colour-ordered five-gluon process yields

\[ A_5(1^\pm, 2^+ , 3^+ , 4^+ , 5^+) = 0 \]
\[ A_5(1^-, 2^- , 3^+ , 4^+ , 5^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \]

In fact, for \( n \) point colour-ordered amplitudes,

\[ A_n(1^\pm , 2^+ , 3^+ , \ldots , n^+) = 0 \]
\[ A_n(1^- , 2^- , 3^+ , \ldots , n^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \]
\[ A_n(1^- , 2^+ , 3^- , \ldots , n^+) = \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \]

Maximally helicity violating (MHV) amplitudes
Gluonic helicity amplitudes

\[ A_n(1^{\pm}, 2^+, 3^+, \ldots, n^+) = 0 \]

effective tree-level supersymmetry
Gluonic helicity amplitudes

\[ A_n(1^-, 2^-, 3^+, \ldots, n^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \]
Witten observed that in twistor space external points lie on certain algebraic curves.

⇒ degree of curve is related to the number of negative helicities and loops.

\[ d = n_- - 1 + l \]
Twistor Space

MHV

NMHV

Twistor inspired developments in perturbative QCD – p.22
MHV rules

Start from on-shell MHV amplitude and define off-shell vertices

\[ V(1^-, 2^-, 3^+, \ldots, n^+, P^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \cdots \langle n - 1 n \rangle \langle n P \rangle \langle P1 \rangle} \]

and

\[ V(1^-, 2^+, 3^+, \ldots, n^+, P^-) = \frac{\langle 1P \rangle^4}{\langle 12 \rangle \cdots \langle n - 1 n \rangle \langle n P \rangle \langle P1 \rangle} \]

Crucial step is off-shell continuation \( P^2 \neq 0 \):

\[ \langle iP \rangle = \frac{\langle i^- | P | \eta^- \rangle}{[P \eta]} = \sum_j \frac{\langle i^- | j | \eta^- \rangle}{[P \eta]} \]

where \( P = \sum_j j \) and \( \eta \) is lightlike auxiliary vector
MHV rules

Must connect up a positive helicity off-shell line to a negative helicity off-shell line with a scalar propagator.

Connecting two MHV’s $\Rightarrow$ amplitude with 3 negative helicities
Connecting three MHV’s $\Rightarrow$ amplitude with 4 negative helicities etc.
Example: six gluon scattering

As an example, let's use the MHV rules to calculate one of the first non-MHV amplitudes

\[ A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) \]

Step 1  Draw all the allowed MHV diagrams
Example: six gluon scattering

There are six MHV graphs

- Graph 1
  - Nodes: 1-, 2-, 3-, 4+, 5+, 6+
  - Connections: 1- to 6+, 2- to 5+, 3- to 4+

- Graph 2
  - Nodes: 1-, 2-, 3-, 4+, 5+, 6+
  - Connections: 1- to 6+, 2- to 5+, 3- to 4+

- Graph 3
  - Nodes: 1-, 2-, 3-, 4+, 5+, 6+
  - Connections: 1- to 6+, 2- to 5+, 3- to 4+

- Graph 4
  - Nodes: 1-, 2-, 3-, 4+, 5+, 6+
  - Connections: 1- to 6+, 2- to 5+, 3- to 4+

- Graph 5
  - Nodes: 1-, 2-, 3-, 4+, 5+, 6+
  - Connections: 1- to 6+, 2- to 5+, 3- to 4+

- Graph 6
  - Nodes: 1-, 2-, 3-, 4+, 5+, 6+
  - Connections: 1- to 6+, 2- to 5+, 3- to 4+
Example: six gluon scattering

Some graphs are not allowed e.g.
Example: six gluon scattering

As an example, let's use the MHV rules to calculate one of the first non-MHV amplitudes

\[ A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) \]

Step 1  Draw all the allowed MHV diagrams
Step 2  Apply MHV rules to each diagram
Example: six gluon scattering: diagram 1

\[
\frac{\langle 12 \rangle^4}{\langle 56 \rangle \langle 61 \rangle \langle 12 \rangle \langle 2 | P | \eta \rangle \langle 5 | P | \eta \rangle} \times \frac{1}{s_{34}} \times \frac{\langle 3 | P | \eta \rangle^4}{\langle 34 \rangle \langle 4 | P | \eta \rangle \langle 3 | P | \eta \rangle}
\]

with \( P = 3 + 4 = -(1 + 2 + 5 + 6) \)
Example: six gluon scattering: diagram 2

\[
\frac{\langle 12 \rangle^4}{\langle 61 \rangle \langle 12 \rangle \langle 2 | P | \eta \rangle \langle 6 | P | \eta \rangle} \times \frac{1}{s_{345}} \times \frac{\langle 3 | P | \eta \rangle^4}{\langle 34 \rangle \langle 45 \rangle \langle 5 | P | \eta \rangle \langle 3 | P | \eta \rangle}
\]

with \( P = 3 + 4 + 5 = -(1 + 2 + 6) \)
Example: six gluon scattering

As an example, let's use the MHV rules to calculate one of the first non-MHV amplitudes

\[ A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) \]

Step 1 Draw all the allowed MHV diagrams
Step 2 Apply MHV rules to each diagram
Step 3 Add up diagrams and check \( \eta \) independence
Next-to MHV amplitude for $n$ gluons

Simplest case: $A_n(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$  $2(n-3)$ graphs

Cachazo, Svrcek and Witten

$$A = \sum_{i=3}^{n-1} \frac{\langle 1| (2, i)| \eta \rangle^3}{\langle (i + 1)| (2, i)| \eta \rangle \langle i + 1 + 2 | \ldots | n \rangle} \frac{1}{s_{2,i}^2} \frac{\langle 23 \rangle^3}{\langle 2| (2, i)| \eta \rangle \langle 34 | \ldots | i | (2, i)| \eta \rangle}
+ \sum_{i=4}^{n} \frac{\langle 12 \rangle^3}{\langle 2| (3, i)| \eta \rangle \langle (i + 1)| (3, i)| \eta \rangle | \ldots | n \rangle} \frac{1}{s_{3,i}^2} \frac{\langle 3| (3, i)| \eta \rangle^3}{\langle 34 | \ldots | i - 1 \rangle \langle i | (3, i)| \eta \rangle}.$$

where $(k, i) = k + \cdots + i$ is the off-shell momentum
⇒ Lorentz invariant and gauge invariant expressions
Generating all the tree amplitudes

Amplitudes with $i-$ and $j+$ helicities

- MHV rules always adds one negative helicity and any number of positive helicities
  $\Rightarrow$ maps out all allowed tree amplitudes
Other processes

MHV rules have been generalised to many other processes

✓ with massless fermions - quarks, gluinos
  Georgiou and Khoze; Wu and Zhu; Georgiou, EWNG and Khoze

✓ with massless scalars - squarks
  Georgiou, EWNG and Khoze; Khoze

✓ with an external Higgs boson
  Dixon, EWNG, Khoze; Badger, EWNG, Khoze

✓ with an external weak boson
  Bern, Forde, Kosower and Mastrolia

Has provided new analytic results for \( n \)-particle amplitudes
Also useful for studying infrared properties of amplitudes
  Birthwright, EWNG, Khoze and Marquard
BCFW on-shell recursion relations

Based on elementary complex analysis - **Cauchy Integral Formula**

\[ \frac{1}{2\pi i} \oint \frac{dz}{z} A(z) = \text{sum of residues} \]

provided that \( A(z) \to 0 \) as \( z \to \infty \)

\[ \text{sum of residues} = A(0) + \ldots \]

Simple enough, but how is this related to scattering amplitudes?
Let's consider an $n$ particle amplitude $A(0)$.

\[ A(0) \rightarrow A(z) = A(0) + A(z) \]

Hatted momenta are shifted to put on-shell:

\[ \hat{i} = i + z\eta, \quad \hat{j} = j - z\eta, \quad \hat{P} = P + z\eta \]

⇒ each vertex is an on-shell amplitude
BCFW recursion relations

- It turns out that the shift $\eta$ is not a momentum, but
  \[ \eta = \lambda_i \bar{\lambda}_j \quad OR \quad \eta = \lambda_j \bar{\lambda}_i \]

- The parameter $z$ is fixed by $\hat{P}^2 = 0$
  \[ z = \frac{P^2}{\langle j|P|i \rangle} \]

- Easy to prove that by complex analysis based on fact that only simple poles in $z$ occur and that $A(z)$ vanishes as $z \to \infty$

  Britto, Cachazo, Feng and Witten

- Requires on-shell three-point vertex contributions - both MHV and $\overline{\text{MHV}}$. 
If we select 3 and 4 to be the special gluons, there are only three diagrams (for any helicities)

For this helicity assignment, the middle diagram is zero!

\[ A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) \]

\[ = \frac{1}{\langle 5|\beta + 4|2 \rangle} \left( \frac{\langle 1|\beta + 3|4 \rangle^3}{[23][34] \langle 56 \rangle \langle 61 \rangle s_{234}} + \frac{\langle 3|\beta + 5|6 \rangle^3}{[61][12] \langle 34 \rangle \langle 45 \rangle s_{345}} \right) \]

Extremely compact analytic results for up to 8 gluons
Other processes

BCF recursion relations have been generalised to other processes

✔ with massless fermions - quarks, gluinos

Luo and Wen

✔ gravitons

Bedford, Brandhuber, Spence and Travaglini; Cachazo and Svrcek

There is nothing (in principle) to stop this approach being applied to particles with mass.

✔ massive coloured scalars

Badger, EWNG, Khoze and Svrcek

✔ massive vector bosons and heavy quarks

Badger, EWNG and Khoze
One loop amplitudes

So far, supersymmetry was not a major factor - tree level amplitudes same for $\mathcal{N} = 4$ $\mathcal{N} = 1$ and QCD

Not true at the loop level due to circulating states

\[
A_{\mathcal{N}=4}^n = A_n^{[1]} + 4A_n^{[1/2]} + 3A_n^{[0]}
\]

\[
A_{\mathcal{N}=1,chiral}^n = A_n^{[1/2]} + A_n^{[0]}
\]

\[
A_{\text{glue}}^n = A_{\mathcal{N}=4}^n - 4A_{\mathcal{N}=1,chiral}^n + A_n^{[0]}
\]

All plus and nearly all-plus amplitudes do not vanish for non-supersymmetric QCD

A lot of progress by a lot of people
One loop amplitudes

Key point is that loop amplitudes contain both poles and cuts - e.g. $\log(x)$ has cut for negative $x$

Cut contributions are fully constructible by using unitarity
- Cut lines are on-shell and 4-dimensional

Pole contributions can be constructed using BCF type recursion and knowledge of factorisation properties

Collectively this is the Unitarity Bootstrap
\( \mathcal{N} = 4 \) and \( \mathcal{N} = 1 \) one-loop amplitudes are constructible from their 4-dimensional cuts
\( \Rightarrow \) employ unitarity techniques

For \( \mathcal{N} = 4 \) all amplitudes are a linear combination of known box integrals

\[
A_n = \sum (a + b + c + d + e + f)
\]
Twistor space interpretation

Coefficients of boxes have very interesting structures.

Bern, Del Duca, Dixon, Kosower; Britto, Cachazo, Feng

\[
\begin{align*}
&c_1^+ (c_1 - 1)^+ + c_2^+ (c_2 + 1)^+ \\
&n^+ (c_2 + 1)^+ \quad c_2^+ \\
&1^- \\
&2^- \\
&3^- \\
&4^+
\end{align*}
\]
Twistor space interpretation

- Four mass box first appears in eight-point amplitude with four negative and four positive helicities

  e.g.

- Still not fully understood
QCD loops

✗ QCD amplitudes more complicated because they are not 4-dimensional cut constructible.
   Rational contribution not probed by 4-d cut

✗ All plus and almost all plus amplitudes no longer zero - but pure rational functions. Not protected by SWI.

✓ Rational parts of infrared divergent amplitudes computed using
  ✓ on-shell recursion relation

Bern, Dixon and Kosower
Recursion relations complicated by double pole terms and boundary terms

✓ Direct Feynman diagram evaluation of rational part

Xiao, Yang, Zhu

✓ \(d\)-dimensional cuts

Anastasiou, Britto, Feng, Kunszt, Mastrolia

Twistor inspired developments in perturbative QCD – p.45
Six gluon amplitude

✓ Analytic computation
Bedford, Berger, Bern, Bidder, Bjerrum-Bohr, Brandhuber, Britto, Buchbinder, Cachazo,
Dixon, Dunbar, Feng, Forde, Kosower, Mastrolia, Perkins, Spence, Travaglini, Xiao, Yang,
Zhu

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>$\mathcal{N} = 4$</th>
<th>$\mathcal{N} = 1$</th>
<th>$\mathcal{N} = 0$ (cut)</th>
<th>$\mathcal{N} = 0$ (rat)</th>
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<tr>
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</table>

✓ Numerical evaluation Ellis, Giele, Zanderighi (06)
Summary - I

On-shell techniques are a very exciting and rapidly developing field

MHV rules for tree-level
Very simple way of deriving \( n \)-point amplitudes for massless partons

BCFW recursion relations for tree-level
Very powerful method for deriving amplitudes for both massless and massive particles

Berends-Giele recursion still looks to be numerically faster

Generalised unitarity and one-loop amplitudes
SUSY amplitudes cut constructible - coefficients of loop integrals can be read off from graphs
QCD amplitudes contain cut-non constructible parts. These simple pole terms can be attacked using the BCFW relations

Or by direct evaluation using Feynman diagrams

Bern, Dixon, Kosower

Xiao, Yang, Zhu
New methods already competitive with traditional methods for loop amplitudes with massless particles - gluons, quarks

Will definitely see all six parton one-loop amplitudes in next few months

Not necessarily the most interesting phenomenologically

Will new methods be useful for amplitudes with heavy particles - top quarks, susy particles, Higgs bosons, vector bosons

In principle heavy particles not a problem - but certainly a complication.

yes for one vector boson plus multiparton e.g. $V + \text{multijet}$

probable for two vector boson plus multiparton e.g. $VV + \text{multijet}$

much more difficult for $pp \rightarrow t\bar{t}b\bar{b}$
1. Predictions for multiparticle final states that occur at high rate and form background to New Physics

High multiplicity, but low order - typically LO or NLO

For example, $pp \rightarrow V + 4$ jets is background to $pp \rightarrow tt$ and other new physics.

2. Precise predictions for hard $pp$ processes involving "standard particles" like $W$, $Z$, jets, top, Higgs, ..

Low multiplicity, but high order - NNLO is emerging standard

For example, Drell Yan cross section.
### State of the Art

<table>
<thead>
<tr>
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<th>$2 \to 1$</th>
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#### NNLO

- ✓ (inclusive) Drell-Yan and Higgs total cross sections – Anastasiou, Dixon, Melnikov, Petriello
- ✓ (inclusive) Drell-Yan and Higgs rapidity distributions – Anastasiou, Dixon, Melnikov, Petriello
- ✓ NNLO evolution – Moch, Vogt, Vermaseren
- ✗ need full set of NNLO observables for global fit. DIS and Drell-Yan will not be enough
Gauge boson production at the LHC

$\sqrt{s} = 14$ TeV
$M = M_Z$
$M/2 \leq \mu \leq 2M$

$pp \rightarrow (Z, \gamma^*) + X$

$\frac{d^2\sigma}{dM/dY} \text{ [pb/GeV]}$

$Y$

$W^-$
$W^+$

$\sqrt{s} = 14$ TeV
$M = M_W$
$M/2 \leq \mu \leq 2M$

$pp \rightarrow W + X$

$\frac{d^2\sigma}{dM/dY} \text{ [pb/GeV]}$

$Y$
Gold-plated process

At LHC NNLO perturbative accuracy better than 1%
⇒ use to determine parton-parton luminosities at the LHC
### State of the Art

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**NNLO ✓** want to calculate $2 \to 2$ to few percent accuracy and use as standard candle to determine PDFs and $\alpha_s$ more accurately

✓ with global pdf fit, gives impact on all observables

✗ still not available
Berends-Giele : Off-shell recursion relations

Full amplitudes can be built up from simpler amplitudes with fewer particles

\[ m \rightarrow m+1 \]

Purple gluons are off-shell, green gluons are on-shell. This is a recursion relation built from off-shell currents.

Particularly suited to numerical solution

Berends, Giele

ALPGEN, HELAC/PHEGAS
Common methods: Colour Ordered Amplitudes

\[ A_n(1, \ldots, n) = \sum_{\text{perms}} Tr(T^{a_1} \ldots T^{a_n}) A_n(1, \ldots, n) \]

Colour-stripped amplitudes \( A_n \): cyclically ordered

Order of external gluons fixed

The subamplitudes \( A_n \) have nice properties in the infrared limits.

Can reconstruct the full amplitude \( A_n \) from \( A_n \).

In the large \( N \) limit,

\[ |A_n(1, \ldots, n)|^2 \sim N^{n-2} \sum_{\text{perms}} |A_n(1, \ldots, n)|^2 \]
Amplitudes in *twistor space* obtained by Fourier transform with respect to positive helicity spinors,

\[ \tilde{\lambda}_{\dot{a}} = i \frac{\partial}{\partial \mu_{\dot{a}}} \]
\[ \mu_{\dot{a}} = i \frac{\partial}{\partial \tilde{\lambda}_{\dot{a}}} \]

Momentum conservation yields

\[ \delta \left( \sum k_j \right) = \int d^4 x \exp \left( i \sum_j x \cdot k_j \right) = \int d^4 x \exp \left( i x^{a_{\dot{a}}} \sum_j \lambda_{ja} \tilde{\lambda}_{j\dot{a}} \right) \]

so that the amplitude in twistor space is

\[ \tilde{A}(\lambda_i, \mu_i) = \int d^4 x \int \prod_i \frac{d^2 \tilde{\lambda}_i}{(2\pi)^2} \exp \left( i \sum_j \left( \mu_{\dot{a}} j + x^{a_{\dot{a}}} \lambda_{ja} \right) \tilde{\lambda}_{j\dot{a}} \right) A(\lambda_i, \tilde{\lambda}_i) \]