THE STANDARD MODEL

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Introduction

An important feature of the Standard Model (SM) is that "it works": it is consistent with, or verified by, all available data, with no compelling evidence for physics beyond.[1] Secondly, it is a unified description, in terms of "gauge theories" of all the interactions of known particles (except gravity). A gauge theory is one that possesses invariance under a set of "local transformations", i.e. transformations whose parameters are space-time dependent.

Electromagnetism is a well-known example of a gauge theory. In this case the gauge transformations are local complex phase transformations of the fields of charged particles, and gauge invariance necessitates the introduction of a massless vector (spin-1) particle, called the photon, whose exchange mediates the electromagnetic interactions.

In the 1950's Yang and Mills considered (as a purely mathematical exercise) extending gauge invariance to include local non-abelian (i.e. non-commuting) transformations such as $SU(2)$. In this case one needs a set of massless vector fields (three in the case of $SU(2)$), which were formally called "Yang-Mills" fields, but are now known as "gauge fields".

In order to apply such a gauge theory to weak interactions, one considers particles which transform into each other under the weak interaction, such as a u-quark and a d-quark, or an electron and a neutrino, to be arranged in doublets of weak isospin. The three gauge bosons are interpreted as the $W^{\pm}$ and Z bosons, that mediate weak interactions in the same way that the photon mediates electromagnetic interactions.

The difficulty in the case of weak interactions was that they are known to be short range, mediated by very massive vector bosons, whereas Yang-Mills fields are required to be massless in order to preserve gauge invariance. The apparent paradox was solved by the application of the "Higgs mechanism". This is a prescription for breaking the gauge symmetry spontaneously. In this scenario one starts with a theory that possesses the required gauge invariance, but where the ground state of the theory is *not* invariant under the gauge transformations. The breaking of the invariance arises in the quantization of the theory, whereas the Lagrangian only contains terms which *are* invariant. One of the consequences of this is that the gauge bosons acquire a mass and the theory can thus be applied to weak interactions.

Spontaneous symmetry breaking and the Higgs mechanism have another extremely important consequence. It leads to a renormalizable theory with massive vector bosons. This means that one can carry out a programme of renormalization in which the infinities that

[1] In saying so we have taken the liberty to allow for neutrino masses (see chapter 7) and discarded some deviations in electroweak precision measurements which are far from conclusive; however, note that there is a 3.4σ deviation between measurement and SM prediction of $g-2$ of the muon, see the remarks in chapter 8.

arise in higher-order calculations can be reabsorbed into the parameters of the Lagrangian (as in the case of QED). Had one simply broken the gauge invariance explicitly by adding mass terms for the gauge bosons, the resulting theory would not have been renormalizable and therefore could not have been used to carry out perturbative calculations. A consequence of the Higgs mechanism is the existence of a scalar (spin-0) particle, the Higgs boson.

The remaining step was to apply the ideas of gauge theories to the strong interaction. The gauge theory of the strong interaction is called "Quantum Chromo Dynamics" (QCD). In this theory the quarks possess an internal property called "colour" and the gauge transformations are local transformations between quarks of different colours. The gauge bosons of QCD are called "gluons" and they mediate the strong interaction.

The union of QCD and the electroweak gauge theory, which describes the weak and electromagnetic interactions, is known as the Standard Model. It has a very simple structure and the different forces of nature are treated in the same fashion, i.e. as gauge theories. It has eighteen fundamental parameters, most of which are associated with the masses of the gauge bosons, the quarks and leptons, and the Higgs. Nevertheless these are not all independent and, for example, the ratio of the W and Z boson masses are (correctly) predicted by the model. Since the theory is renormalizable, perturbative calculations can be performed at higher order that predict cross sections and decay rates for both strongly and weakly interacting processes. These predictions, when confronted with experimental data, have been confirmed very successfully. As both predictions and data are becoming more and more precise, the tests of the Standard Model are becoming increasingly stringent.

1 QED as an Abelian Gauge Theory

The aim of this lecture is to start from a symmetry of the fermion Lagrangian and show that "gauging" this symmetry (= making it well behaved) implies classical electromagnetism with its gauge invariance, the $e\bar{e}\gamma$ interaction, and that the photon must be massless.

1.1 Preliminaries

In the Field Theory lectures at this school, the quantum theory of an interacting scalar field was introduced, and the voyage from the Lagrangian to the Feynman rules was made. Fermions can be quantised in a similar way, and the propagators one obtains are the Green functions for the Dirac wave equation (the inverse of the Dirac operator) of the QED/QCD course. In this course, I will start from the Lagrangian (as opposed to the wave equation) of a free Dirac fermion, and add interactions, to construct the Standard Model Lagrangian in classical field theory. That is, the fields are treated as functions, and I will not discuss creation and annihiliation operators. However, to extract Feynman rules from the Lagrangian, I will implicitly rely on the rules developed for scalar fields in the Field Theory course.

1.2 Gauge Transformations

Consider the Lagrangian density for a free Dirac field ψ:

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi \tag{1.1}$$

This Lagrangian density is invariant under a phase transformation of the fermion field

$$\psi \;\rightarrow\; e^{iQ\omega}\psi, \qquad \overline{\psi} \;\rightarrow\; e^{iQ\omega}\overline{\psi}, \tag{1.2}$$

where Q is the charge operator ($Q\psi = +\psi, Q\bar{\psi} = -\bar{\psi}$), ω is a real constant (i.e. independent of x) and $\overline{\psi}$ is the conjugate field.

The set of all numbers $e^{-i\omega}$ form a group[2]. This particular group is "abelian" which is to say that any two elements of the group commute. This just means that

$$e^{-i\omega_1}e^{-i\omega_2} \;=\; e^{-i\omega_2}e^{-i\omega_1}. \tag{1.3}$$

[2]A group is a mathematical term for a set, where multiplication of elements is defined and results in another element of the set. Furthermore, there has to be a 1 element (s.t. $1 \times a = a$) and an inverse (s.t. $a \times a^{-1} = 1$) for each element a of the set.

This particular group is called $U(1)$ which means the group of all unitary 1×1 matrices. A unitary matrix satisfies $U^+ = U^{-1}$ with U^+ being the adjoint matrix.

We can now state the invariance of the Lagrangian eq. (1.1) under phase transformations in a more fancy way by saying that the Lagrangian is invariant under global $U(1)$ transformations. By global we mean that ω does not depend on x.

For the purposes of these lectures it will usually be sufficient to consider infinitesimal group transformations, i.e. we assume that the parameter ω is sufficiently small that we can expand in ω and neglect all but the linear term. Thus we write

$$e^{-i\omega} = 1 \; - \; i\,\omega \; + \; \mathcal{O}(\omega^2). \tag{1.4}$$

Under such infinitesimal phase transformations the field ψ changes according to

$$\psi \to \psi + \delta\psi = \psi + \; i\,Q\omega\,\psi, \tag{1.5}$$

and the conjugate field $\overline{\psi}$ by

$$\overline{\psi} \to \overline{\psi} + \delta\overline{\psi} \; = \overline{\psi} + \, i\,Q\omega\,\overline{\psi} = \overline{\psi} - \; i\,\omega\,\overline{\psi}, \tag{1.6}$$

such that the Lagrangian density remains unchanged (to order ω).

At this point we should note that global transformations are not very attractive from a theoretical point of view. The reason is that making the same transformation at every space-time point requires that all these points 'know' about the transformation. But if I were to make a certain transformation at the top of Mont Blanc, how can a point somewhere in England know about it? It would take some time for a signal to travel from the Alps to England.

Thus, we have two options at this point. Either, we simply note the invariance of eq. (1.1) under global $U(1)$ transformations and put this aside as a curiosity, or we insist that invariance under gauge transformations is a fundamental property of nature. If we take the latter option we have to require invariance under local transformations. Local means that the parameter of the transformation, ω, now depends on the space-time point x. Such local (i.e. space-time dependent) transformations are called "gauge transformations".

If the parameter ω depends on the space-time point then the field ψ transforms as follows under infinitesimal transformations

$$\delta\psi(x) = i\,\omega(x)\,\psi(x); \qquad \delta\overline{\psi}(x) = -i\,\omega(x)\,\overline{\psi}(x). \tag{1.7}$$

Note that the Lagrangian density eq. (1.1) now is *no longer* invariant under these transformations, because of the partial derivative between $\overline{\psi}$ and ψ. This derivative will act on the space-time dependent parameter $\omega(x)$ such that the Lagrangian density changes by an amount $\delta\mathcal{L}$, where

$$\delta\mathcal{L} \;=\; -\overline{\psi}(x)\,\gamma^\mu\,[\partial_\mu Q\omega(x)]\,\psi(x). \tag{1.8}$$

The square brackets in $[\partial_\mu Q\omega(x)]$ are introduced to indicate that the derivative ∂_μ acts only inside the brackets. It turns out that we can restore gauge invariance if we assume that the fermion field interacts with a vector field A_μ, called a "gauge field", with an interaction term

$$-\,e\,\overline{\psi}\,\gamma^\mu A_\mu Q\psi \tag{1.9}$$

added to the Lagrangian density which now becomes

$$\mathcal{L} = \overline{\psi}\;\left(i\gamma^\mu\left(\partial_\mu + i\,e\,QA_\mu\right) - m\right)\psi. \tag{1.10}$$

In order for this to work we must also assume that apart from the fermion field transforming under a gauge transformation according to eq. (1.7) the gauge field, A_μ, also changes according to

$$-eQA_\mu \rightarrow -eQ(A_\mu + \delta A_\mu(x)) \;=\; -eQ\;A_\mu + Q\partial_\mu\,\omega(x). \tag{1.11}$$

So $\delta A_\mu(x) = -Q\partial_\mu\,\omega(x)/e$.

> **Exercise 1.1**
> Using eqs. (1.7) and (1.11) show that under a gauge transformation $\delta(-e\,\overline{\psi}\,\gamma^\mu A_\mu\psi) \;=\; -\overline{\psi}(x)\,\gamma^\mu\,[\partial_\mu Q\omega(x)]\,\psi(x)$.

This change exactly cancels with eq. (1.8), so that once this interaction term has been added the gauge invariance is restored. We recognize eq. (1.10) as being the fermionic part of the Lagrangian density for QED, where e is the electric charge of the fermion and A_μ is the photon field.

In order to have a proper quantum field theory, in which we can expand the photon field A_μ in terms of creation and annihilation operators for photons, we need a kinetic term for the photon, i.e. a term which is quadratic in the derivative of the field A_μ. Without such a term the Euler-Lagrange equation for the gauge field would be an algebraic equation and we could use it to eliminate the gauge field altogether from the Lagrangian. We need to ensure that in introducing a kinetic term we do not spoil the invariance under gauge transformations. This is achieved by defining the field strength tensor, $F_{\mu\nu}$, as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.12}$$

where the derivative is understood to act on the A-field only.[3] It is easy to see that under the gauge transformation eq. (1.11) each of the two terms on the right hand side of eq. (1.12)

[3]Strictly speaking we should therefore write $F_{\mu\nu} = [\partial_\mu A_\nu] - [\partial_\nu A_\mu]$; you will find that the brackets are often omitted.

change, but the changes cancel out. Thus we may add to the Lagrangian any term which depends on $F_{\mu\nu}$ (and which is Lorentz invariant, thus, with all Lorentz indices contracted). Such a term is $aF_{\mu\nu}F^{\mu\nu}$. This gives the desired term which is quadratic in the derivative of the field A_μ. If we choose the constant a to be $-1/4$ then the Lagrange equations of motion match exactly (the relativistic formulation of) Maxwell's equations.[4]

We have thus arrived at the Lagrangian density for QED, but from the viewpoint of demanding invariance under $U(1)$ gauge transformations rather than starting with Maxwell's equations and formulating the equivalent quantum field theory.

The Lagrangian density for QED is:

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\,\left(i\gamma^\mu\left(\partial_\mu + i\,e\,QA_\mu\right) - m\right)\psi. \tag{1.13}$$

Exercise 1.2
Starting with the Lagrangian density for QED write down the Euler-Lagrange equations for the gauge field A_μ and show that this results in Maxwell's equations.

In the Field Theory lectures, we have seen that a term $\lambda\phi^4$ in the Lagrangian gave $4!\lambda$ as the coupling of four ϕs in perturbation theory. Neglecting the combinatoric factors, it is plausible that eq. (1.13) gives the $\gamma\bar{e}e$ Feynman Rule used in the QED course, $-ie\gamma^\mu$, for negatively charged particles.

Note that we are *not* allowed to add a mass term for the photon. A term such as $M^2A_\mu A^\mu$ added to the Lagrangian density is not invariant under gauge transformations as it would lead to

$$\delta\mathcal{L} \;=\; \frac{2M^2}{e}A^\mu(x)\partial_\mu\omega(x) \neq 0. \tag{1.14}$$

Thus the masslessness of the photon can be understood in terms of the requirement that the Lagrangian be gauge invariant.

1.3 Covariant Derivatives

Before leaving the abelian case, it is useful to introduce the concept of a "covariant derivative". This is not essential for abelian gauge theories, but will be an invaluable tool when we extend these ideas to non-abelian gauge theories.

[4]The determination of this constant a is the *only* place that a match to QED has been used. The rest of the Lagrangian density is obtained purely from the requirement of local $U(1)$ invariance. A different constant would simply mean a different normalization of the photon field.

The covariant derivative D_μ is defined to be

$$D_\mu \equiv \partial_\mu + i\,e\,A_\mu. \tag{1.15}$$

It has the property that given the transformations of the fermion field eq. (1.7) and the gauge field eq. (1.11) the quantity $D_\mu\psi$ transforms in the same way under gauge transformations as ψ.

Exercise 1.3
Show that under an infinitesimal gauge transformation $D_\mu\psi$ transforms as $D_\mu\psi \to D_\mu\psi + \delta(D_\mu\psi)$ with $\delta(D_\mu\psi) = -i\omega(x)D_\mu\psi$.

We may thus rewrite the QED Lagrangian density as

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi. \tag{1.16}$$

Furthermore the field strength $F_{\mu\nu}$ can be expressed in terms of the commutator of two covariant derivatives, i.e.

$$\begin{aligned} F_{\mu\nu} \;&=\; -\frac{i}{e}\,[D_\mu, D_\nu] = \;-\frac{i}{e}[\partial_\mu,\partial_\nu] + [\partial_\mu, A_\nu] + [A_\mu, \partial_\nu] + i\,e\,[A_\mu, A_\nu] \\ &=\; \partial_\mu A_\nu - \partial_\nu A_\mu, \end{aligned} \tag{1.17}$$

where in the last line we have adopted the conventional notation again and left out the square brackets. Notice that when using eq. (1.17) the derivatives act only on the A-field.

1.4 Gauge Fixing

The guiding principle of this chapter has been to hold onto the $U(1)$ symmetry. This forced us to introduce a new massless field A_μ which we could interpret as the photon. In this subsection we will try to quantise the photon field (e.g. calculate its propagator) by naively following the prescription used for scalars and fermions, which will not work. This should not be surprising, because A_μ has four real components, introduced to maintain gauge symmetry. However the physical photon has two polarisation states. This difficulty can be resolved by "fixing the gauge" (breaking our precious gauge symmetry) in the Lagrangian in such a way as to maintain the gauge symmetry in observables.[5]

[5]The gauge symmetry is also preserved in the Path Integral, which is a sum over all field configurations weighted by $\exp\{i\int \mathcal{L}d^4x\}$. In path integral quantisation, which is an alternative to the canonical approach used in the Field Theory lectures, Green functions are calculated from the path integral and it is unimportant that the gauge symmetry seems broken in the Lagrangian.

In general, if the part of the action that is quadratic in some field $\phi(x)$ is given in terms of the Fourier transform $\tilde{\phi}(p)$ by

$$S_\phi = \int d^4p\, \tilde{\phi}(-p)\mathcal{O}(p)\tilde{\phi}(p), \tag{1.18}$$

then the propagator for the field ϕ may be written as

$$i\,\mathcal{O}^{-1}(p). \tag{1.19}$$

In the case of QED the part of the Lagrangian that is quadratic in the photon field is given by $-1/4\, F^{\mu\nu}F_{\mu\nu} = -1/2\, A^\mu \left(-g_{\mu\nu}\partial^\sigma\partial_\sigma + \partial_\mu\partial_\nu\right) A^\nu$, where we have used partial integration to obtain the second expression. In momentum space, the quadratic part of the action is then given by

$$S_A = \int d^4p\, \frac{1}{2}\tilde{A}^\mu(-p)\left(-g_{\mu\nu}\, p^2 + p_\mu p_\nu\right)\tilde{A}^\nu(p). \tag{1.20}$$

Unfortunately the operator $(-g_{\mu\nu}\, p^2 + p_\mu p_\nu)$ does not have an inverse. This can be most easily seen by noting $(-g_{\mu\nu}\, p^2 + p_\mu p_\nu)\, p^\nu = 0$. This means that the operator $(-g_{\mu\nu}\, p^2 + p_\mu p_\nu)$ has an eigenvector (p^ν) with eigenvalue 0 and is therefore not invertible. Thus it seems we are not able to write down the propagator of the photon. We solve this problem by adding to the Lagrangian density a gauge fixing term

$$-\frac{1}{2(1-\xi)}\left(\partial_\mu A^\mu\right)^2. \tag{1.21}$$

With this term included (again in momentum space), S_A becomes

$$S_A = \int d^4p\, \frac{1}{2}\tilde{A}^\mu(-p)\left(-g_{\mu\nu}\, p^2 - \frac{\xi}{1-\xi}p_\mu p_\nu\right)\tilde{A}^\nu(p), \tag{1.22}$$

and, noting the relation

$$\left(g_{\mu\nu}\, p^2 + \frac{\xi}{1-\xi}p_\mu p_\nu\right)\left(g^{\nu\rho} - \xi\frac{p^\nu p^\rho}{p^2}\right) = p^2 g_\mu{}^\rho, \tag{1.23}$$

we see that the propagator for the photon may now be written as

$$-i\left(g_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2}. \tag{1.24}$$

The special choice $\xi = 0$ is known as the Feynman gauge. In this gauge the propagator eq. (1.24) is particularly simple and we will use it most of the time.

This procedure of gauge fixing seems strange: first we worked hard to get a gauge invariant Lagrangian, and then we spoil gauge invariance by introducing a gauge fixing term.

The point is that we have to fix the gauge in order to be able to perform a calculation. Once we have computed a physical quantity, the dependence on the gauge cancels. In other words, it does not matter how we fix the gauge, and in particular, what value for ξ we take. The choice $\xi = 0$ is simply a matter of convenience. A more careful procedure would be to leave ξ arbitrary and check that all ξ-dependence in the final result cancels. This gives us a strong check on the calculation, however, at the price of making the computation much more tedious.

The procedure of fixing the gauge in order to be able to perform a calculation, even though the final result does not depend on how we have fixed the gauge, can be understood by the following analogy. Assume we wanted to calculate some scalar quantity (say the time it takes for a point mass to get from one point to another) in our ordinary 3-dimensional Euclidean space. To do so, we choose a coordinate system, perform the calculation and get our final result. Of course, the result does not depend on how we choose the coordinate system, but in order to be able to perform the calculation we have to fix it somehow. Picking a coordinate system corresponds to fixing a gauge, and the independence of the result on the coordinate system chosen corresponds to the gauge invariance of physical quantities. To take this one step further we remark that not all quantities are independent of the coordinate system. For example, the x-coordinate of the position of the point mass at a certain time depends on our choice. Similarly, there are important quantities that are gauge dependent. One example is the gauge boson propagator given in eq. (1.24). However, all measurable quantities (observables) are gauge invariant. This is where our analogy breaks down: in our Euclidean example there are measurable quantities that do depend on the choice of the coordinate system.

Finally we should mention that eq. (1.21) is by far not the only way to fix the gauge but it will be sufficient for these lectures to consider gauges defined through eq. (1.21). These gauges are called covariant gauges.

1.5 Summary

- It is possible for the Lagrangian for a (complex) Dirac field to be invariant under local $U(1)$ transformations (phase rotations), in which the phase parameter depends on space-time. In order to accomplish this we include an interaction with a vector gauge boson which transforms under the local (gauge) transformation according to eq. (1.11).

- This interaction is encoded by replacing the derivative ∂_μ by the covariant derivative D_μ defined by eq. (1.15). $D_\mu \psi$ transforms under gauge transformations as $e^{-i\omega} D_\mu \psi$.

- The kinetic term for the gauge boson is $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu}$ is proportional to the commutator $[D_\mu, D_\nu]$ and is invariant under gauge transformations.

- The gauge boson must be massless, since a term proportional to $A_\mu A^\mu$ is *not* invariant under gauge transformations and hence not included in the Lagrangian.
- The resulting Lagrangian is identical to that of QED.
- In order to define the propagator we have to specify a certain gauge; the resulting gauge dependence cancels in physical observables.

2 Non-Abelian Gauge Theories

In this lecture, the "gauge" concept will be constructed so that the gauge bosons have self-interactions — as are observed among the gluons of QCD, and the $W^{\pm}$, Z and γ of the electroweak sector. However, the gauge bosons will still be massless. (We will see how to give the $W^{\pm}$ and Z their observed masses in the Higgs chapter.)

2.1 Global Non-Abelian Transformations

We apply the ideas of the previous lecture to the case where the transformations do not commute with each other, i.e. the group is "non-abelian".

Consider n free fermion fields $\{\psi_i\}$, arranged in a multiplet ψ:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ . \\ . \\ \psi_n \end{pmatrix} \tag{2.1}$$

for which the Lagrangian density is

$$\begin{aligned} \mathcal{L} &= \overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi, \\ &\equiv \overline{\psi}^{\,i}\,(i\gamma^\mu\partial_\mu - m)\,\psi_i, \end{aligned} \tag{2.2}$$

where the index i is summed from 1 to n. Eq. (2.2) is therefore a shorthand for

$$\mathcal{L} \;=\; \overline{\psi}^{\,1}\,(i\gamma^\mu\partial_\mu - m)\,\psi_1 + \overline{\psi}^{\,2}\,(i\gamma^\mu\partial_\mu - m)\,\psi_2 + \ldots\,. \tag{2.3}$$

The Lagangian density (2.2) is invariant under (space-time *independent*) complex rotations in ψ_i space:

$$\psi \to \mathbf{U}\psi, \qquad \bar{\psi} \to \bar{\psi}\mathbf{U}^\dagger, \tag{2.4}$$

where $\mathbf{U}$ is an $n \times n$ matrix such that

$$\mathbf{U}\mathbf{U}^\dagger = 1, \qquad \det[\mathbf{U}] = 1. \tag{2.5}$$

The transformation (2.4) is called an internal symmetry, which rotates the fields (e.g. quarks of different colour) among themselves.

The group of matrices satisfying the conditions (2.5) is called $SU(n)$. This is the group of special, unitary $n \times n$ matrices. Special in this context means that the determinant is equal to 1. In order to specify an $SU(n)$ matrix completely we need n^2-1 real parameters. Indeed, we need $2n^2$ real parameters to determine an arbitrary complex $n \times n$ matrix. But there are n^2 constraints due to the unitary requirements and one additional constraint due to the requirement det $= 1$.

An arbitrary $SU(n)$ matrix can be written as

$$\mathbf{U} = e^{-i\sum_{a=1}^{n^2-1}\omega^a\,\mathbf{T}^a} \equiv e^{-i\omega^a\,\mathbf{T}^a} \tag{2.6}$$

where we again have adopted Einstein's summation convention. The ω^a, $a \in \{1\dots n^2-1\}$, are real parameters, and the $\mathbf{T}^a$ are called the generators of the group.

> **Exercise 2.1**
> Show that the unitarity of the $SU(n)$ matrices entails hermiticity of the generators and that the requirement of det $= 1$ means that the generators have to be traceless.

In the case of $U(1)$ there was just one generator. Here we have n^2-1 generators $\mathbf{T}^a$. There is still some freedom left of how to normalize the generators. We will adopt the usual normalization convention

$$\mathrm{tr}(\mathbf{T}^a\mathbf{T}^b) = \frac{1}{2}\,\delta_{ab}. \tag{2.7}$$

The reason we can always enforce eq. (2.7) is that $\mathrm{tr}(\mathbf{T}^a\mathbf{T}^b)$ is a real matrix symmetric in $a \leftrightarrow b$. Thus it can be diagonalized. If you have problems getting on friendly terms with the concept of generators, for the moment you can think of them as traceless, hermitian $n \times n$ matrices. (This is, however, not the complete picture.)

The crucial new feature of the group $SU(n)$ is that two elements of $SU(n)$ generally do not commute, i.e.

$$e^{-i\omega_1^a\,\mathbf{T}^a}e^{-i\omega_2^b\,\mathbf{T}^b} \neq e^{-i\omega_2^b\,\mathbf{T}^b}e^{-i\omega_1^a\,\mathbf{T}^a} \tag{2.8}$$

(compare to eq. (1.3)). To put this in a different way, the group algebra is not trivial. For the commutator of two generators we have

$$[\mathbf{T}^a, \mathbf{T}^b] \equiv i f^{abc}\mathbf{T}^c \neq 0 \tag{2.9}$$

where we defined the structure constants of the group, f^{abc}, and used the summation convention again. The structure constants are totally antisymmetric. This can be seen as follows: from eq. (2.9) it is obvious that $f^{abc} = -f^{bac}$. To convince us of the antisymmetry in the other indices as well, we note that multiplying eq. (2.9) by $\mathbf{T}^d$ and taking the trace, using eq. (2.7), we get $1/2\,if^{abd} = \mathrm{tr}(\mathbf{T}^a\mathbf{T}^b\mathbf{T}^d) - \mathrm{tr}(\mathbf{T}^b\mathbf{T}^a\mathbf{T}^d) = \mathrm{tr}(\mathbf{T}^a\mathbf{T}^b\mathbf{T}^d) - \mathrm{tr}(\mathbf{T}^a\mathbf{T}^d\mathbf{T}^b)$.

2.2 Non-Abelian Gauge Fields

Now suppose we allow the transformation U to depend on space-time. Then the Lagrangian density changes by $\delta\mathcal{L}$ under this "non-abelian gauge transformation", where

$$\delta\mathcal{L} \;=\overline{\psi}\,\mathbf{U}^{\dagger}\gamma^{\mu}\left(\partial_{\mu}\mathbf{U}\right)\psi. \tag{2.10}$$

The local gauge symmetry can be restored by introducing a covariant derivative $\mathbf{D}_{\mu}$, giving interactions with gauge bosons, such that

$$\mathbf{D}_{\mu}\mathbf{U}(x)\psi(x) = \mathbf{U}(x)\mathbf{D}_{\mu}\psi(x). \tag{2.11}$$

This is like the electromagnetic case, except that $\mathbf{D}_{\mu}$ is now a matrix,

$$i\mathbf{D}_{\mu} = i\mathbf{I}\partial_{\mu} - g\mathbf{A}_{\mu} \tag{2.12}$$

where $\mathbf{A}_{\mu} = \mathbf{T}^{a}A^{a}_{\mu}$. It contains $n^2 - 1$ vector (spin one) gauge bosons, A^{a}_{μ}, one for each generator of $SU(n)$. Under a gauge transformation U, $\mathbf{A}_{\mu}$ should transform as

$$\mathbf{A}_{\mu} \rightarrow \mathbf{U}\mathbf{A}_{\mu}\mathbf{U}^{\dagger} + \frac{i}{g}\left(\partial_{\mu}\mathbf{U}\right)\mathbf{U}^{\dagger}. \tag{2.13}$$

This ensures that the Lagrangian density

$$\mathcal{L} \;=\; \overline{\psi}\;\left(i\gamma^{\mu}\mathbf{D}_{\mu} - m\right)\psi \tag{2.14}$$

is invariant under local $SU(n)$ gauge transformations. It can be checked that eq. (2.13) reduces to the gauge transformation of electromagnetism in the abelian limit.

Exercise 2.2

(For algebraically ambitious people): perform an infinitesimal gauge transformation on $\psi, \bar{\psi}$ and $\mathbf{D}$, using (2.6), and show that to linear order in the ω_a, $\bar{\psi}\gamma_{\mu}\mathbf{D}^{\mu}\psi$ is invariant.

Exercise 2.3

Show that in the $SU(2)$ case, the covariant derivative is

$$i\mathbf{D}_{\mu} = \begin{pmatrix} i\partial_{\mu} - \frac{g}{2}W^3_{\mu} & -\frac{g}{2}(W^1_{\mu} - iW^2_{\mu}) \\ -\frac{g}{2}(W^1_{\mu} + iW^2_{\mu}) & i\partial_{\mu} + \frac{g}{2}W^3_{\mu} \end{pmatrix},$$

and find the usual charged current interactions for the lepton doublet

$$\psi = \begin{pmatrix} \nu \\ e \end{pmatrix}$$

by defining $W^{\pm} = (W^1 \mp iW^2)/\sqrt{2}$.

Exercise 2.4
Include the $U(1)$ hypercharge interaction in the previous question; show that the covariant derivative acting on the lepton doublet (of hypercharge $Y = -1/2$) is

$$i\mathbf{D}_\mu = \begin{pmatrix} i\partial_\mu - \frac{g}{2}W_\mu^3 - g'YB_\mu & -\frac{g}{2}(W_\mu^1 - iW_\mu^2) \\ -\frac{g}{2}(W_\mu^1 + iW_\mu^2) & i\partial_\mu + \frac{g}{2}W_\mu^3 - g'YB_\mu \end{pmatrix}.$$

Define

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}$$

and write the diagonal (neutral) interactions in terms of Z_μ and A_μ. Extract $\sin\theta_W$ in terms of g and g'. (Recall that the photon does not interact with the neutrino.)

The kinetic term for the gauge bosons is again constructed from the field strengths $F^a_{\mu\nu}$ which are defined from the commutator of two covariant derivatives,

$$\mathbf{F}_{\mu\nu} = -\frac{i}{g}\left[\mathbf{D}_\mu, \mathbf{D}_\nu\right], \tag{2.15}$$

where the matrix $\mathbf{F}_{\mu\nu}$ is given by

$$\mathbf{F}_{\mu\nu} = \mathbf{T}^a F^a_{\mu\nu}, \tag{2.16}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc}\, A^b_\mu A^c_\nu. \tag{2.17}$$

Notice that $\mathbf{F}_{\mu\nu}$ is gauge *variant*, unlike the $U(1)$ case. We know the transformation of $\mathbf{D}$ from (2.13), so

$$[\mathbf{D}_\mu, \mathbf{D}_\nu] \to \mathbf{U}\,[\mathbf{D}_\mu, \mathbf{D}_\nu]\,\mathbf{U}^\dagger. \tag{2.18}$$

The gauge invariant kinetic term for the gauge bosons is therefore

$$-\frac{1}{2}\mathbf{Tr}\ \mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu}, \tag{2.19}$$

where the trace is in $SU(n)$ space, and summation over the index a is implied.

In sharp contrast with the abelian case, this term does not only contain terms which are quadratic in the derivatives of the gauge boson fields, but also the terms

$$g\, f^{abc}(\partial_\mu A^a_\nu)A^b_\mu A^c_\nu - \frac{1}{4}g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu. \tag{2.20}$$

This means that there is a very important difference between abelian and non-abelian gauge theories. For non-abelian gauge theories the gauge bosons interact with each other via both three-point and four-point interaction terms. The three point interaction term contains a derivative, which means that the Feynman rule for the three-point vertex involves the momenta of the particles going into the vertex. We shall write down the Feynman rules in detail later.

Once again, a mass term for the gauge bosons is forbidden, since a term proportional to $A_\mu^a A^{a\,\mu}$ is *not* invariant under gauge transformations.

2.3 Gauge Fixing

As in the case of QED, we need to add a gauge-fixing term in order to be able to derive a propagator for the gauge bosons. In Feynman gauge this means adding the term $-\frac{1}{2}(\partial^\mu A_\mu^a)^2$ to the Lagrangian density, and the propagator (in momentum space) becomes

$$-i\,\delta_{ab}\,\frac{g_{\mu\nu}}{p^2}.$$

There is one unfortunate complication, which is mentioned briefly here for the sake of completeness, although one only needs to know about it for the purpose of performing higher loop calculations with non-abelian gauge theories:

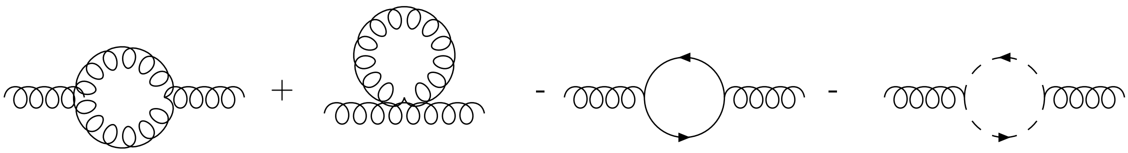
If one goes through the formalism of gauge-fixing carefully, it turns out that at higher orders extra loop diagrams emerge. These diagrams involve additional particles that are mathematically equivalent to interacting scalar particles and are known as a "Faddeev-Popov ghosts". For each gauge field there is such a ghost field. These are *not* to be interpreted as physical scalar particles which could in principle be observed experimentally, but merely as part of the gauge-fixing programme. For this reason they are referred to as "ghosts". Furthermore they have two peculiarities:

1. They only occur inside loops. This is because they are not really particles and cannot occur in initial or final states, but are introduced to clean up a difficulty that arises in the gauge-fixing mechanism.

2. They behave like fermions even though they are scalars (spin zero). This means that we need to count a minus sign for each loop of Faddeev-Popov ghosts in any Feynman diagram.

We shall display the Feynman rules for these ghosts later.

Thus, for example, the Feynman diagrams which contribute to the one-loop corrections to the gauge boson propagator are



(a) (b) (c) (d)

Diagram (a) involves the three-point interaction between the gauge bosons, diagram (b) involves the four-point interaction between the gauge bosons, diagram (c) involves a loop of fermions, and diagram (d) is the extra diagram involving the Faddeev-Popov ghosts. Note that both diagrams (c) and (d) have a minus sign in front of them because both fermions and Faddeev-Popov ghosts obey Fermi statistics.

2.4 The Lagrangian for a General Non-Abelian Gauge Theory

Let us summarize what we have found so far: Consider a gauge group $\mathcal{G}$ of "dimension" N (for $SU(n): \ N \equiv n^2-1$), whose N generators, $\mathbf{T}^a$, obey the commutation relations $\left[\mathbf{T}^a, \mathbf{T}^b\right] = if_{abc}\mathbf{T}^c$, where f_{abc} are called the "structure constants" of the group.

The Lagrangian density for a gauge theory with this group, with a fermion multiplet ψ_i, is given (in Feynman gauge) by

$$\mathcal{L} \ = \ -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + i\overline{\psi}\left(\gamma^\mu \mathbf{D}_\mu - m\mathbf{I}\right)\psi - \frac{1}{2}(\partial^\mu A^a_\mu)^2 + \mathcal{L}_{\rm FP} \tag{2.21}$$

where

$$F^a_{\mu\nu} \ = \ \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc}A^b_\mu A^c_\nu, \tag{2.22}$$

$$\mathbf{D}_\mu \ = \ \partial_\mu \mathbf{I} + i\,g\,\mathbf{T}^a A^a_\mu \tag{2.23}$$

and

$$\mathcal{L}_{\rm FP} \ = \ -\xi^a\partial^\mu\partial_\mu\eta^a + g\, f_{acb}\xi^a A^c_\mu(\partial^\mu\eta^b). \tag{2.24}$$

Under an infinitesimal gauge transformation the N gauge bosons A^a_μ change by an amount that contains a term which is not linear in A^a_μ:

$$\delta A^a_\mu(x) \ = -f^{abc}A^b_\mu(x)\omega^c(x) + \frac{1}{g}\partial_\mu\omega^a(x), \tag{2.25}$$

whereas the field strengths $F^a_{\mu\nu}$ transform by a change

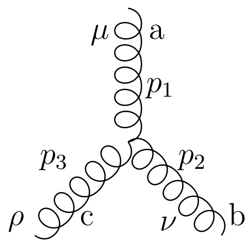
$$\delta F^a_{\mu\nu}(x) \ = -f^{abc}\,F^b_{\mu\nu}(x)\,\omega^c. \tag{2.26}$$

In other words, they transform as the "adjoint" representation of the group (which has as many components as there are generators). This means that the quantity $F^a_{\mu\nu} F^{a\,\mu\nu}$ (summation over a, μ, ν implied) is invariant under gauge transformations.

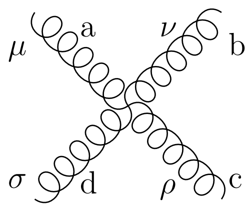
2.5 Feynman Rules

The Feynman rules for such a gauge theory can be read off directly from the Lagrangian. As mentioned previously, the propagators are obtained by taking all terms bilinear in the field and inverting the corresponding operator (and multiplying by i). The rules for the vertices are obtained by simply taking (i times) the factor which multiplies the corresponding term in the Lagrangian. The explicit rules are given in the following.

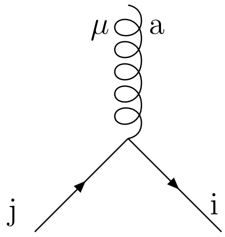
Vertices:
(Note that all momenta are defined as flowing into the vertex!)



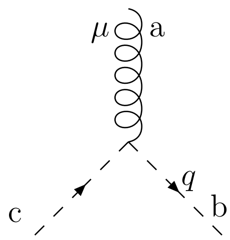
$$-g\, f^{abc} \left(g_{\mu\nu} \left(p_1 - p_2\right)_\rho + g_{\nu\rho} \left(p_2 - p_3\right)_\mu + g_{\rho\mu} \left(p_3 - p_1\right)_\nu \right)$$



$$-i\, g^2 f^{eab} f^{ecd} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right)$$
$$-i\, g^2 f^{eac} f^{ebd} \left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho}\right)$$
$$-i\, g^2 f^{ead} f^{ebc} \left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}\right)$$

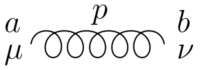


$$-i\, g\, \gamma^\mu \left(T^a\right)_{ij}$$

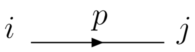


$$g\, f^{abc}\, q_\mu$$

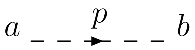
Propagators:



Gluon: $-i\,\delta_{ab}\,g_{\mu\nu}/p^2$



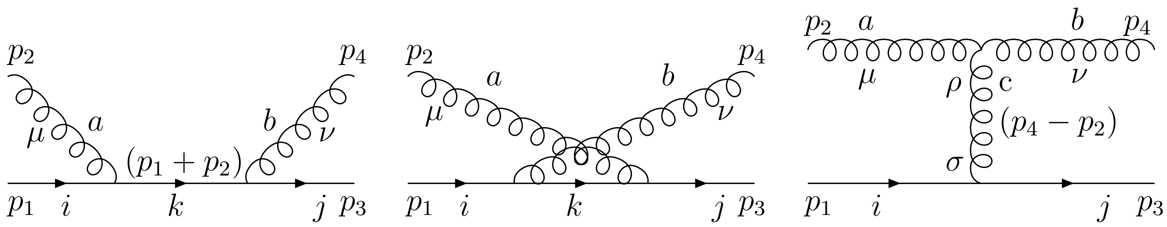
Fermion: $i\,\delta_{ij}(\gamma^\mu p_\mu + m)/(p^2 - m^2)$



Faddeev-Popov ghost: $i\,\delta_{ab}/p^2$

2.6 An Example

As an example of the application of these Feynman rules, we consider the process of Compton scattering, but this time for the scattering of non-abelian gauge bosons and fermions, rather than photons. We need to calculate the amplitude for a gauge boson of momentum p_2 and gauge label a to scatter off a fermion of momentum p_1 and gauge label i producing a fermion of momentum p_3 and gauge label j and a gauge boson of momentum p_4 and gauge label b. Note that $i, j \in \{1 \ldots n\}$ whereas $a, b \in \{1 \ldots n^2 - 1\}$. In addition to the two Feynman diagrams one gets in the QED case there is a third diagram involving the self-interaction of the gauge bosons.



(a) (b) (c)

We will assume that the fermions are massless (i.e. that we are at sufficiently high energies so that we may neglect their masses), and work in terms of the Mandelstam variables

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (p_3 + p_4)^2, \\
t &= (p_1 - p_3)^2 = (p_2 - p_4)^2, \\
u &= (p_1 - p_4)^2 = (p_2 - p_3)^2.
\end{aligned}
$$

The polarizations are accounted for by contracting the amplitude obtained for the above diagrams with the polarization vectors $\epsilon^\mu(\lambda_2)$ and $\epsilon^\nu(\lambda_4)$. Each diagram consists of two vertices and a propagator and so their contributions can be read off from the Feynman rules.

For diagram (a) we get

$$\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}^j(p_3)\left(-i\,g\,\gamma^\nu(\mathbf{T}^b)^k_j\right)\left(i\,\frac{\gamma\cdot(p_1+p_2)}{s}\right)\left(-i\,g\,\gamma^\mu(\mathbf{T}^a)^i_k\right)u_i(p_1)$$

$$= -i\,\frac{g^2}{s}\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}\,(p_3)\left(\gamma^\nu\gamma\cdot(p_1+p_2)\gamma^\mu\right)\left(\mathbf{T}^b\mathbf{T}^a\right)u(p_1).$$

For diagram (b) we get

$$\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}^j(p_3)\left(-i\,g\,\gamma^\mu(\mathbf{T}^a)^k_j\right)\left(i\,\frac{\gamma\cdot(p_1-p_4)}{u}\right)\left(-i\,g\,\gamma^\nu(\mathbf{T}^b)^i_k\right)u_i(p_1)$$

$$= -i\,\frac{g^2}{u}\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}\,(p_3)\left(\gamma^\nu\gamma\cdot(p_1-p_4)\gamma^\mu\right)\left(\mathbf{T}^a\mathbf{T}^b\right)u(p_1).$$

Note that here the order of the $\mathbf{T}$ matrices is the other way around compared to diagram (a).

Diagram (c) involves the three-point gauge-boson self-coupling. Since the Feynman rule for this vertex is given with incoming momenta, it is useful to replace the outgoing gauge-boson momentum p_4 by $-p_4$ and understand this to be an incoming momentum. Note that the internal gauge-boson line carries momentum $p_4 - p_2$ coming into the vertex. The three incoming momenta that are to be substituted into the Feynman rule for the vertex are therefore p_2, $-p_4$, $p_4 - p_2$. The vertex thus becomes

$$-g\,f_{abc}\left(g_{\mu\nu}(p_2+p_4)_\rho + g_{\rho\nu}(p_2-2p_4)_\mu + g_{\mu\rho}(p_4-2p_2)_\nu\right),$$

and the diagram gives

$$\epsilon^\mu(\lambda_2)\epsilon^\nu(\lambda_4)\overline{u}^j(p_3)\left(-i\,g\,\gamma_\sigma(\mathbf{T}^c)^i_j\right)u_i(p_1)\left(-i\frac{g^{\rho\sigma}}{t}\right)$$

$$\times\;\left(-g\,f_{abc}\right)\left(g_{\mu\nu}(p_2+p_4)_\rho + g_{\rho\nu}(p_2-2p_4)_\mu + g_{\mu\rho}(p_4-2p_2)_\nu\right)$$

$$= -i\frac{g^2}{t}\epsilon^\mu(\lambda_2)\epsilon^\nu(\lambda_4)\overline{u}\,(p_3)\left[\mathbf{T}^a,\mathbf{T}^b\right]\gamma^\rho u(p_1)\left(g_{\mu\nu}\,(p_2+p_4)_\rho - 2(p_4)_\mu g_{\nu\rho} - 2(p_2)_\nu g_{\mu\rho}\right),$$

where in the last step we have used the commutation relation eq. (2.9) and the fact that the polarization vectors are transverse so that $p_2\cdot\epsilon(\lambda_2)=0$ and $p_4\cdot\epsilon(\lambda_4)=0$.

Exercise 2.4
Draw all the Feynman diagrams for the tree level amplitude for two gauge bosons with momenta p_1 and p_2 to scatter into two gauge bosons with momenta q_1 and q_2. Label the momenta of the external gauge boson lines.

2.7 Summary

- A non-abelian gauge theory is one in which the Lagrangian is invariant under local transformations of a non-abelian group.
- This invariance is achieved by introducing a gauge boson for each generator of the group. The partial derivative in the Lagrangian for the fermion field is replaced by a covariant derivative as defined in eq. (2.23).
- The gauge bosons transform under infinitesimal gauge transformations in a non-linear way given by eq. (2.25).
- The field strengths, $F^a_{\mu\nu}$, are obtained from the commutator of two covariant derivatives and are given by eq. (2.22). They transform as the adjoint representation under gauge transformations such that the quantity $F^a_{\mu\nu}F^{a\,\mu\nu}$ is invariant.
- $F^a_{\mu\nu}F^{a\,\mu\nu}$ contains terms which are cubic and quartic in the gauge bosons, indicating that these gauge bosons interact with each other.
- The gauge-fixing mechanism leads to the introduction of Faddeev-Popov ghosts which are scalar particles that occur only inside loops and obey Fermi statistics.

3 Quantum Chromodynamics

Quantum Chromodynamics (QCD) is the theory of the strong interaction. It is nothing but a non-abelian gauge theory with the group $SU(3)$. Thus, the quarks are described by a field ψ_i where i runs from 1 to 3. The quantum number associated with the label i is called colour. The eight gauge bosons which have to be introduced in order to preserve local gauge invariance are the eight 'gluons'. These are taken to be the carriers which mediate the strong interaction in the same way that photons are the carriers which mediate the electromagnetic interactions.

The Feynman rules for QCD are therefore simply the Feynman rules listed in the previous lecture, with the gauge coupling constant, g, taken to be the strong coupling, g_s, (more about this later), the generators $\mathbf{T}^a$ taken to be the eight generators of $SU(3)$ in the triplet representation, and f^{abc}, a, b, c, $= 1 \dots 8$ are the structure constants of $SU(3)$ (you can look them up in a book but normally you will not need their explicit form).

Thus we now have a quantum field theory which can be used to describe the strong interaction.

3.1 Running Coupling

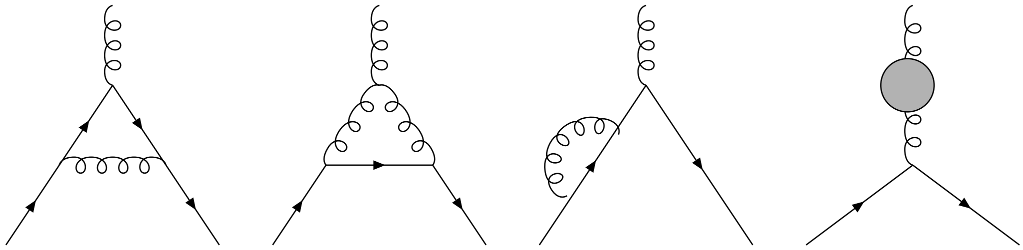
The coupling for the strong interaction is the QCD gauge coupling, g_s. We usually work in terms of α_s defined as

$$\alpha_s = \frac{g_s^2}{4\pi}. \tag{3.1}$$

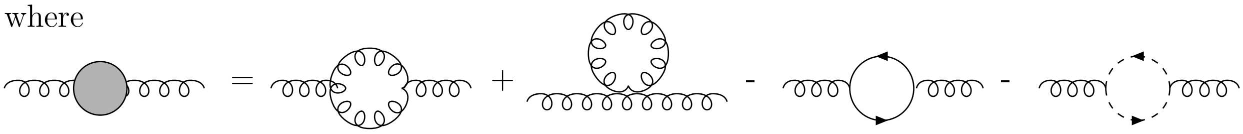
Since the interactions are strong, we would expect α_s to be too large to perform reliable calculations in perturbation theory. On the other hand the Feynman rules are only useful within the context of perturbation theory.

This difficulty is resolved when we understand that 'coupling constants' are not constant at all. The electromagnetic fine structure constant, α, has the value $1/137$ only at energies which are not large compared to the electron mass. At higher energies it is larger than this. For example, at LEP energies it takes a value close to $1/129$. In contrast to QED, it turns out that in the non-abelian gauge theories of the Standard Model the weak and the strong coupling *decrease* as the energy increases.

To see how this works within the context of QCD we note that when we perform higher order perturbative calculations there are loop diagrams which have the effect of 'dressing' the couplings. For example, the one-loop diagrams which dress the coupling between a quark and a gluon are:



where



are the diagrams needed to calculate the one-loop corrections to the gluon propagator.

These diagrams contain UV divergences and need to be renormalized, e.g. by subtracting at some renormalization scale μ. This scale then appears inside a logarithm for the renormalized quantities. This means that if the squared momenta of all the external particles coming into the vertex are of order Q^2, where $Q \gg \mu$, then the above diagrams give rise to a correction which contains a logarithm of the ratio Q^2/μ^2:

$$-\alpha_s^2 \beta_0 \ln\left(Q^2/\mu^2\right). \tag{3.2}$$

This correction is interpreted as the correction to the effective QCD coupling, $\alpha_s(Q^2)$, at momentum scale Q, i.e.

$$\alpha_s(Q^2) \;= \alpha_s(\mu^2) \;-\; \alpha_s(\mu^2)^2 \beta_0 \ln\left(Q^2/\mu^2\right) \;+\; \ldots . \tag{3.3}$$

The coefficient β_0 is calculated to be

$$\beta_0 \;=\; \frac{11\, N_c \;-\; 2\, n_f}{12\,\pi}, \tag{3.4}$$

where N_c is the number of colours (=3), n_f is the number of active flavours, i.e. the number of flavours whose mass threshold is below the momentum scale Q. Note that β_0 is *positive*, which means that the coefficient in front of the logarithm in eq. (3.3) is *negative*, so that the effective coupling *decreases* as the momentum scale is increased.

A more precise analysis shows that the effective coupling obeys the differential equation

$$\frac{\partial\,\alpha_s(Q^2)}{\partial\,\ln(Q^2)} \;=\; \beta\left(\alpha_s(Q^2)\right), \tag{3.5}$$

where β has the perturbative expansion

$$\beta(\alpha_s) \;=\; -\beta_0\,\alpha_s^2 - \beta_1\,\alpha_s^3 + \mathcal{O}(\alpha_s^4) + \ldots . \tag{3.6}$$

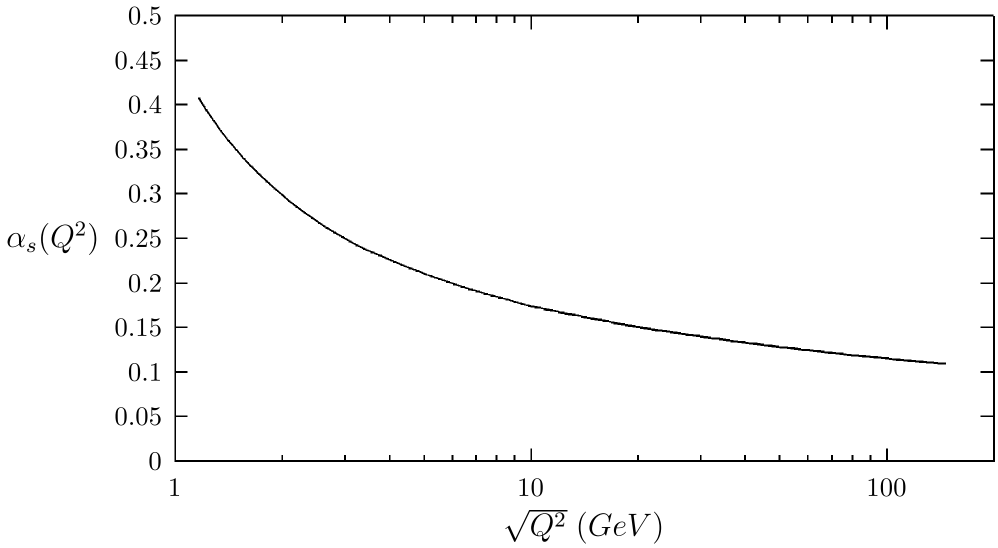


Figure 3.1: The running of $\alpha_s(Q^2)$ with β taken to two loops.

In order to solve this differential equation we need a boundary value. Nowadays this is usually taken to be the measured value of the coupling at scale of the Z boson mass, $M_Z = 91.19$ GeV, which is measured to be

$$\alpha_s(M_Z^2) \;=\; 0.118 \pm 0.002\,. \tag{3.7}$$

This is one of the free parameters of the Standard Model.[6]

The running of $\alpha_s(Q^2)$ is shown in figure 3.1. We can see that for momentum scales above about 2 GeV the coupling is less than 0.3 so that one can hope to carry out reliable perturbative calculations for QCD processes with energy scales larger than this.

Gauge invariance requires that the gauge coupling for the interaction between gluons must be exactly the same as the gauge coupling for the interaction between quarks and gluons. The β-function could therefore have been calculated from the higher order corrections to the three-gluon (or four-gluon) vertex and must yield the same result, despite the fact that it is calculated from a completely different set of diagrams.

[6]Previously the solution to eq. (3.5) (to leading order) was written as $\alpha_s(Q^2) = 4\pi/\beta_0 \ln(Q^2/\Lambda_{\rm QCD}^2)$ and the scale $\Lambda_{\rm QCD}$ was used as the standard parameter which sets the scale for the magnitude of the strong coupling. This turns out to be rather inconvenient since it needs to be adjusted every time higher order corrections are taken into consideration and the number of active flavours has to be specified. The detour via $\Lambda_{\rm QCD}$ also introduces additional truncation errors and can complicate the error analysis.

Exercise 3.1
Draw the Feynman diagrams needed for the calculation of the one-loop correction to the triple gluon coupling (don't forget the Faddeev-Popov ghost loops).

Exercise 3.2
Solve equation (3.5) using β to leading order only, and calculate the value of α_s at a momentum scale of 10 GeV. Use the value at M_Z given by eq. (3.7). Calculate also the error in α_s at 10 GeV.

3.2 Quark (and Gluon) Confinement

This argument can be inverted to provide an answer to the question of why we have never seen quarks or gluons in a laboratory. Asymptotic Freedom tells us that the effective coupling between quarks becomes weaker at shorter distances (equivalent to higher energies/momentum scales). Conversely it implies that the effective coupling grows as we go to larger distances. Therefore, the complicated system of gluon exchanges which leads to the binding of quarks (and antiquarks) inside hadrons leads to a stronger and stronger binding as we attempt to pull the quarks apart. This means that we can never isolate a quark (or a gluon) at large distances since we require more and more energy to overcome the binding as the distance between the quarks grows. Instead, when the energy contained in the 'string' of bound gluons and quarks becomes large enough, the colour-string breaks and more quarks are created, leaving more colourless hadrons, but no isolated, coloured quarks.

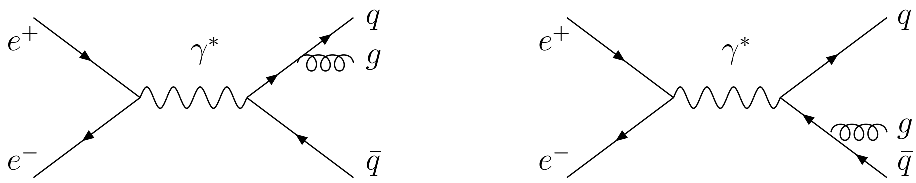
The upshot of this is that the only free particles which can be observed at macroscopic distances from each other are colour singlets. This mechanism is known as "quark confinement". The details of how it works are not fully understood. Nevertheless the argument presented here is suggestive of such confinement and at the level of non-perturbative field theory, lattice calculations have confirmed that for non-abelian gauge theories the binding energy does indeed grow as the distance between quarks increases.[7]

Thus we have two different pictures of the world of strong interactions: On one hand, at sufficiently short distances, which can be probed at sufficiently large energies, we can consider quarks and gluons (partons) interacting with each other. In this regime we can perform calculations of the scattering cross sections between quarks and gluons (called the "partonic hard cross section") in perturbation theory because the running coupling is sufficiently

[7]Lattice QCD simulations have also succeeded in calculating the spectrum of many observed hadrons and also hadronic matrix elements for certain processes from 'first principles', i.e. without using perturbative expansions or phenomenological models.

small. On the other hand, before we can make a direct comparison with what is observed in accelerator experiments, we need to take into account the fact that the quarks and gluons bind (hadronize) into colour singlet hadrons, and it is only these colour singlet states that are observed directly. The mechanism for this hadronization is beyond the scope of perturbation theory and not understood in detail. Nevertheless Monte Carlo programs have been developed which simulate the hadronization in such a way that the results of the short-distance perturbative calculations at the level of quarks and gluons can be confronted with experiments measuring hadrons in a successful way.

Thus, for example, if we wish to calculate the cross section for an electron-positron annihilation into three jets (at high energies), we first calculate, in perturbation theory, the process for electron plus positron to annihilate into a virtual photon (or Z boson) which then decays into a quark and antiquark, and an emitted gluon. At leading order the two Feynman diagrams for this process are:[8]



However, before we can compare the results of this perturbative calculation with experimental data on three jets of observed hadrons, we need to perform a convolution of this calculated cross section with a Monte Carlo simulation that accounts for the way in which the final state partons (quarks and gluons) bind with other quarks and gluons to produce observed hadrons. It is only after such a convolution has been performed that one can get a reliable comparison of the calculated observables (like cross sections or event shapes) with data.

Likewise, if we want to calculate scattering processes including initial state hadrons we need to account for the probability of finding a particular quark or gluon inside an initial hadron with a given fraction of the initial hadron's momentum (these are called "parton distribution functions").

> **Exercise 3.3**
> Draw the (tree level) Feynman diagrams for the process $e^+e^- \to 4$jets. Consider only one photon exchange plus the QCD contributions (do not include Z boson exchange or WW production).

[8]The contraction of the one loop diagram (where a gluon connects the quark and antiquark) with the $e^+e^- \to q\bar{q}$ amplitude is of the same order α_s and has to be taken into account to get an infra-red finite result. However, it does not lead to a three-jet event (on the partonic level).

3.3 θ-Parameter of QCD

There is one more gauge invariant term that can be written down in the QCD Lagrangian:

$$\mathcal{L}_\theta = \theta \frac{g_s^2}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}. \tag{3.8}$$

Here $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor (in four dimensions). Since we should work with the most general gauge invariant Lagrangian there is no reason to omit this term. However, adding this term to the Lagrangian leads to a problem, called the "strong CP problem".

To understand the nature of the problem, we first convince ourselves that this term violates CP. In QED we would have

$$\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \mathbf{E} \cdot \mathbf{B}, \tag{3.9}$$

and for QCD we have a similar expression except that $\mathbf{E}^a$ and $\mathbf{B}^a$ carry a colour index — they are known as the chromoelectric and chromomagnetic fields. Under charge conjugation both the electric and magnetic field change sign. But under parity the electric field, which is a proper vector, changes sign, whereas the magnetic field, which is a polar vector, does not change sign. Thus we see that the term $\mathbf{E} \cdot \mathbf{B}$ is odd under CP.

For this reason, the parameter θ in front of this term must be exceedingly small in order not to give rise to strong interaction contributions to CP violating quantities such as the electric dipole moment of the neutron. The current experimental limits on this dipole moment tell us that $\theta < 10^{-10}$. Thus we are tempted to think that θ is zero. Nevertheless, strictly speaking θ is a free parameter of QCD, and is sometimes considered to be the nineteenth free parameter of the Standard Model.

Of course we simply could set θ to zero (or a very small number) and be happy with it.[9] However, whenever a free parameter is zero or extremely small, we would like to understand the reason. The fact that we do not know why this term is absent (or so small) is the strong CP problem.

There are several possible solutions to the strong CP problem that offer explanations as to why this term is absent (or small). One possible solution is through imposing an additional symmetry, leading to the postulation of a new, hypothetical, weakly interacting particle, called the "(Peccei-Quinn) axion". Unfortunately none of these solutions have been confirmed yet and the problem is still unresolved.

Another question is why is this not a problem in QED? In fact a term like eq. (3.8) can also

[9] To be precise, setting $\theta \to 0$ in the Lagrangian would not be enough, as $\theta \neq 0$ can also be generated through higher order electroweak radiative corrections, requiring a fine-tuning beyond $\theta \to 0$.

be written down in QED. A thorough discussion of this point is beyond the scope of this lecture. Suffice to say that this term can be written (in QED and QCD) as a total divergence, so it seems that it can be eliminated from the Lagrangian altogether. However, in QCD (but not in QED) there are non-perturbative effects from the non-trivial topological structure of the vacuum (somewhat related to so called "instantons" you probably have heard about) which prevent us from neglecting the θ-term.

3.4 Summary

- Quarks transform as a triplet representation of colour $SU(3)$ (each quark can have one of three colours).
- The eight gauge bosons of QCD are the gluons which are the carriers that mediate the strong interaction.
- The coupling of quarks to gluons (and gluons to each other) decreases as the energy scale increases. Therefore, at high energies one can perform reliable perturbative calculations for strongly interacting processes.
- As the distance between quarks increases the binding increases, such that it is impossible to isolate individual quarks or gluons. The only observable particles are colour singlet hadrons. Perturbative calculations performed at the quark and gluon level must be supplemented by accounting for the recombination of final state quarks and gluons into observed hadrons as well as the probability of finding these quarks and gluons inside the initial state hadrons (if applicable).
- QCD admits a gauge invariant strong CP violating term with a coefficient θ. This parameter is known to be very small from limits on CP violating phenomena such as the electric dipole moment of the neutron.

4 Spontaneous Symmetry Breaking

We have seen that in an unbroken gauge theory the gauge bosons must be massless. This is exactly what we want for QED (massless photon) and QCD (massless gluons). However, if we wish to extend the ideas of describing interactions by a gauge theory to the weak interactions, the symmetry must somehow be broken since the carriers of the weak interactions (W and Z bosons) are massive (weak interactions are very short range). We could simply break the symmetry by hand by adding a mass term for the gauge bosons, which we know violates the gauge symmetry. However, this would destroy renormalizability of our theory.

Renormalizable theories are preferred because they are more predictive. As discussed in the Field Theory and QED lectures, there are divergent results (infinities) in QED and QCD, and these are said to be renormalizable theories. So what could be worse about a non-renormalizable theory? The critical issue is the number of divergences: few in a renormalizable theory, and infinite in the non-renormalizable case. Associated to every divergence is a parameter that must be extracted from data, so renormalizable theories can make testable predictions once a few parameters are measured. For instance, in QCD, the coupling g_s has a divergence. But once α_s is measured in one process, the theory can be tested in other processes.[10]

In this chapter we will discuss a way to give masses to the W and Z, called "spontaneous symmetry breaking", which maintains the renormalizability of the theory. In this scenario the Lagrangian maintains its symmetry under a set of local gauge transformations. On the other hand, the lowest energy state, which we interpret as the vacuum (or ground state), is *not* a singlet of the gauge symmetry. There is an infinite number of states each with the same ground-state energy and nature chooses one of these states as the 'true' vacuum.

4.1 Massive Gauge Bosons and Renormalizability

In this subsection we will convince ourselves that simply adding by hand a mass term for the gauge bosons will destroy the renormalizability of the theory. It will not be a rigorous argument, but will illustrate the difference between introducing mass terms for the gauge bosons in a brute force way and introducing them via spontaneous symmetry breaking.

Higher order (loop) corrections generate ultraviolet divergences. In a renormalizable theory,

[10]It should be noted that effective field theories, though formally not renormalizable, can nevertheless be very valuable as they often allow for a simplified description of a more 'complete' or fundamental theory in a resticted energy range. Popular examples are Chiral Perturbation Theory, Heavy Quark Effective Theory and Non-Relativistic QCD.

these divergences can be absorbed into the parameters of the theory we started with, and in this way can be 'hidden'. As we go to higher orders we need to absorb more and more terms into these parameters, but there are only as many divergent quantities as there are parameters. So, for instance, in QED the Lagrangian we start with contains the fermion field, the gauge boson field, and interactions whose strength is controlled by e and m. Being a renormalizable theory, all divergences of diagrams can be absorbed into these quantities (irrespective of the number of loops or legs), and once e and m are measured, all other observables (cross sections, $g-2$, etc.) can be predicted.

In order to ensure that this programme can be carried out there have to be restrictions on the allowed interaction terms. Furthermore all the propagators have to decrease like $1/p^2$ as the momentum $p \to \infty$. Note that this is how the massless gauge-boson propagator eq. (1.24) behaves. If these conditions are not fulfilled, then the theory generates more and more divergent terms as one calculates to higher orders, and it is not possible to absorb these divergences into the parameters of the theory. Such theories are said to be "non-renormalizable".

Now we can convince ourselves that simply adding a mass term $M^2 A_\mu A^\mu$ to the Lagrangian given in eq. (2.21) will lead to a non-renormalizable theory. To start with we note that such a term will modify the propagator. Collecting all terms bilinear in the gauge fields in momentum space we get (in Feynman gauge)

$$\frac{1}{2} A_\mu \left(-g^{\mu\nu}(p^2 - M^2) + p^\mu p^\nu \right) A_\nu . \tag{4.1}$$

We have to invert this operator to get the propagator which now takes the form

$$\frac{i}{p^2 - M^2} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2} \right) . \tag{4.2}$$

Note that this propagator, eq. (4.2), has a much worse ultraviolet behavior in that it goes to a constant for $p \to \infty$. Thus, it is clear that the ultraviolet properties of a theory with a propagator as given in eq. (4.2) are worse than for a theory with a propagator as given in eq. (1.24). According to our discussion at the beginning of this subsection we conclude that without the explicit mass term $M^2 A_\mu A^\mu$ the theory is renormalizable, whereas with this term it is not. In fact, it is precisely the gauge symmetry that ensures renormalizability. Breaking this symmetry results in the loss of renormalizability.

The aim of spontaneous symmetry breaking is to break the gauge symmetry in a more subtle way, such that we can still give the gauge bosons a mass but retain renormalizability.

4.2 Spontaneous Symmetry Breaking

Spontaneous symmetry breaking is a phenomenon that is by far not restricted to gauge symmetries. It is a subtle way to break a symmetry by still requiring that the Lagrangian remains invariant under the symmetry transformation. However, the ground state of the symmetry is *not* invariant, i.e. *not* a singlet under a symmetry transformation.

In order to illustrate the idea of spontaneous symmetry breaking, consider a pen that is completely symmetric with respect to rotations around its axis. If we balance this pen on its tip on a table, and start to press on it with a force precisely along the axis we have a perfectly symmetric situation. This corresponds to a Lagrangian which is symmetric (under rotations around the axis of the pen in this case). However, if we increase the force, at some point the pen will bend (and eventually break). The question then is in which direction will it bend. Of course we do not know, since all directions are equal. But the pen will pick one and by doing so it will break the rotational symmetry. This is spontaneous symmetry breaking.

A better example can be given by looking at a point mass in a potential

$$V(\vec{r}) \;=\; \mu^2 \vec{r}\cdot\vec{r} + \lambda(\vec{r}\cdot\vec{r})^2. \tag{4.3}$$

This potential is symmetric under rotations and we assume $\lambda > 0$ (otherwise there would be no stable ground state). For $\mu^2 > 0$ the potential has a minimum at $\vec{r} = 0$, thus the point mass will simply fall to this point. The situation is more interesting if $\mu^2 < 0$. For two dimensions the potential is shown in Fig. 4.1. If the point mass sits at $\vec{r} = 0$ the system is not in the ground state but the situation is completely symmetric. In order to reach the ground state, the symmetry has to be broken, i.e. if the point mass wants to roll down, it has to decide in which direction. Any direction is equally good, but one has to be picked. This is exactly what spontaneous symmetry breaking means. The Lagrangian (here the potential) is symmetric (here under rotations around the z-axis), but the ground state (here the position of the point mass once it rolled down) is not. Let us formulate this in a slightly more mathematical way for gauge symmetries. We denote the ground state by $|0\rangle$. A spontaneously broken gauge theory is a theory whose Lagrangian is invariant under gauge transformations, which is exactly what we have done in chapters 1 and 2. The new feature in a spontaneously broken theory is that the ground state is not invariant under gauge transformations. This means

$$e^{-i\omega^a \mathbf{T}^a}\,|0\rangle \;\neq\; |0\rangle \tag{4.4}$$

which entails

$$\mathbf{T}^a\,|0\rangle \;\neq\; 0 \qquad \text{for some} \;\; a. \tag{4.5}$$

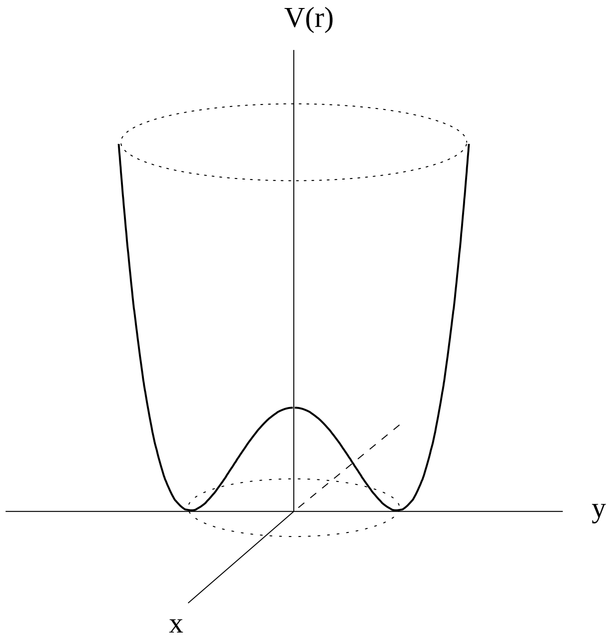


Figure 4.1: A potential that leads to spontaneous symmetry breaking.

Eq. (4.5) follows from eq. (4.4) upon expansion in ω^a. Thus, the theory is spontaneously broken if there exists at least one generator that does not annihilate the vacuum.

In the next section we will explore the concept of spontaneous symmetry breaking in the context of gauge symmetries in more detail, and we will see that, indeed, this way of breaking the gauge symmetry has all the desired features.

4.3 The Abelian Higgs Model

For simplicity, we will start by spontaneously breaking the $U(1)$ gauge symmetry in a theory of one complex scalar field. In the Standard Model, it will be a non-abelian gauge theory that is spontaneously broken, but all the important ideas can simply be translated from the $U(1)$ case considered here.

The Lagrangian density for a gauged complex scalar field, with a mass term and a quartic self-interaction, may be written as

$$\mathcal{L} \;=\; (D_\mu\Phi)^* D^\mu\Phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\Phi), \tag{4.6}$$

where the potential $V(\Phi)$, is given by

$$V(\Phi) \;=\; \mu^2\Phi^*\Phi + \lambda\left|\Phi^*\Phi\right|^2, \tag{4.7}$$

and the covariant derivative D_μ and the field-strength tensor $F_{\mu\nu}$ are given in eqs. (1.15) and (1.12) respectively. This Lagrangian is invariant under $U(1)$ gauge transformations

$$\Phi \;\rightarrow\; e^{-i\omega(x)}\Phi. \tag{4.8}$$

Provided μ^2 is positive this potential has a minimum at $\Phi = 0$. We call the $\Phi = 0$ state the vacuum and expand Φ in terms of creation and annihilation operators that populate the higher energy states. In terms of a quantum field theory, where Φ is an operator, the precise statement is that the operator Φ has zero vacuum expectation value, i.e. $\langle 0|\Phi|0\rangle = 0$.

Now suppose we *reverse* the sign of μ^2, so that the potential becomes

$$V(\Phi) \;=\; -\mu^2\Phi^*\Phi + \lambda\left|\Phi^*\Phi\right|^2, \tag{4.9}$$

with $\mu^2 > 0$. We see that this potential no longer has a minimum at $\Phi = 0$, but a (local) *maximum*. The minimum occurs at

$$\Phi = e^{i\theta}\sqrt{\frac{\mu^2}{2\lambda}} \equiv e^{i\theta}\frac{v}{\sqrt{2}}, \tag{4.10}$$

where θ can take any value from 0 to 2π. There is an infinite number of states each with the same lowest energy, i.e. we have a degenerate vacuum. The symmetry breaking occurs in the choice made for the value of θ which represents the true vacuum. For convenience we shall choose $\theta = 0$ to be our vacuum. Such a choice constitutes a spontaneous breaking of the $U(1)$ invariance, since a $U(1)$ transformation takes us to a different lowest energy state. In other words the vacuum breaks $U(1)$ invariance. In quantum field theory we say that the field Φ has a non-zero vacuum expectation value

$$\langle\Phi\rangle \;=\; \frac{v}{\sqrt{2}}. \tag{4.11}$$

But this means that there are 'excitations' with zero energy, that take us from the vacuum to one of the other states with the same energy. The only particles which can have zero energy are massless particles (with zero momentum). We therefore expect a massless particle in such a theory.

To see that we do indeed get a massless particle, let us expand Φ around its vacuum expectation value,

$$\Phi \;=\; \frac{e^{i\phi/v}}{\sqrt{2}}\left(\frac{\mu}{\sqrt{\lambda}} + H\right) \simeq \frac{1}{\sqrt{2}}\left(\frac{\mu}{\sqrt{\lambda}} + H + i\phi\right). \tag{4.12}$$

The fields H and ϕ have zero vacuum expectation values and it is these fields that are expanded in terms of creation and annihilation operators of the particles that populate the excited states. Of course, it is the H-field that corresponds to the Higgs field.

We now want to write the Lagrangian in terms of the H and ϕ fields. In order to get the potential we insert eq. (4.12) into eq. (4.9) and find

$$V = \mu^2 H^2 + \mu\sqrt{\lambda}\left(H^3 + \phi^2 H\right) + \frac{\lambda}{4}\left(H^4 + \phi^4 + 2H^2\phi^2\right) + \frac{\mu^4}{4\lambda}. \qquad (4.13)$$

Note that in eq. (4.13) there is a mass term for the H-field, $\mu^2 H^2 \equiv M_H/2H^2$, where we have defined[11]

$$M_H = \sqrt{2}\mu. \qquad (4.14)$$

However, there is *no* mass term for the field ϕ. Thus ϕ is a field for a massless particle called the "Goldstone boson". We will look at this issue in a more general way in section 4.4. Next let us consider the kinetic term. We plug eq. (4.12) into $(D_\mu\Phi)^* D^\mu\Phi$ and get

$$\begin{aligned}(D_\mu\Phi)^* D^\mu\Phi &= \frac{1}{2}\partial_\mu H\partial^\mu H + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}g^2v^2A_\mu A^\mu + \frac{1}{2}g^2A_\mu A^\mu(H^2+\phi^2)\\ &- gA_\mu\left(\phi\partial_\mu H - H\partial_\mu\phi\right) + gvA_\mu\partial^\mu\phi + g^2vA_\mu A^\mu H. \qquad (4.15)\end{aligned}$$

There are several important features in eq. (4.15). Firstly, the gauge boson has acquired a mass term $1/2g^2v^2A_\mu A^\mu \equiv 1/2M_A^2A_\mu A^\mu$, where we have defined

$$M_A = gv. \qquad (4.16)$$

Secondly, there is a coupling of the gauge field to the H-field,

$$g^2vA_\mu A^\mu H = gM_AA_\mu A^\mu H. \qquad (4.17)$$

It is important to remember that this coupling is proportional to the mass of the gauge boson. Finally, there is also the bilinear term $g\,v\,A^\mu\partial_\mu\phi$, which after integrating by parts (for the action S) may be written as $-\,M_A\,\phi\,\partial_\mu A^\mu$. This mixes the Goldstone boson, ϕ, with the longitudinal component of the gauge boson, with strength M_A (when the gauge-boson field A_μ is separated into its transverse and longitudinal components, $A_\mu = A_\mu^L + A_\mu^T$, where $\partial^\mu A_\mu^T = 0$). Later on, we will use the gauge freedom to get rid of this mixing term.

4.4 Goldstone Bosons

In the previous subsection we have seen that there is a massless boson, called the Goldstone boson, associated with the flat direction in the potential. Goldstone's theorem describes the appearance of massless bosons when a global (not gauge) symmetry is spontaneously broken.

[11]Note that for a real field ϕ representing a particle of mass m the mass term is $\frac{1}{2}m^2\phi^2$, whereas for a complex field the mass term is $m^2\phi^\dagger\phi$.

Suppose we have a theory whose Lagrangian is invariant under a symmetry group $\mathcal{G}$ with N generators $\mathbf{T}^a$ and the symmetry group of the vacuum forms a subgroup $\mathcal{H}$ of $\mathcal{G}$, with m generators. This means that the vacuum state is still invariant under transformations generated by the m generators of $\mathcal{H}$, but not the remaining $N-m$ generators of the original symmetry group $\mathcal{G}$. Thus we have

$$\begin{aligned} \mathbf{T}^a|0\rangle &= 0 \qquad a = 1\ldots m, \\ \mathbf{T}^a|0\rangle &\neq 0 \qquad a = m+1\ldots N. \end{aligned} \tag{4.18}$$

Goldstone's theorem states that there will be $N-m$ massless particles (one for each broken generator of the group). The case considered in this section is special in that there is only one generator of the symmetry group (i.e. $N=1$) which is broken by the vacuum. Thus, there is no generator that leaves the vacuum invariant (i.e. $m=0$) and we get $N-m=1$ Goldstone boson.

Like all good general theorems, Goldstone's theorem has a loophole, which arises when one considers a gauge theory, i.e. when one allows the original symmetry transformations to be local. In a spontaneously broken gauge theory, the choice of which vacuum is the true vacuum is equivalent to choosing a gauge, which is necessary in order to be able to quantize the theory. What this means is that the Goldstone bosons, which can, in principle, transform the vacuum into any of the states degenerate with the vacuum, now affect transitions into states which are not consistent with the original gauge choice. This means that the Goldstone bosons are "unphysical" and are often called "Goldstone ghosts".

On the other hand the quantum degrees of freedom associated with the Goldstone bosons are certainly there *ab initio* (before a choice of gauge is made). What happens to them? A massless vector boson has only two degrees of freedom (the two directions of polarization of a photon), whereas a massive vector (spin-one) particle has three possible values for the helicity of the particle. In a spontaneously broken gauge theory, the Goldstone boson associated with each broken generator provides the third degree of freedom for the gauge bosons. This means that the gauge bosons become massive. The Goldstone boson is said to be "eaten" by the gauge boson. This is related to the mixing term between A_L^μ and ϕ of the previous subsection. Thus, in our abelian model, the two degrees of freedom of the complex field Φ turn out to be the Higgs field and the longitudinal component of the (now massive) gauge boson. There is no physical, massless particle associated with the degree of freedom ϕ present in Φ.

4.5 The Unitary Gauge

As mentioned above, we want to use the gauge freedom to choose a gauge such that there are no mixing terms between the longitudinal component of the gauge field and the Goldstone boson. Recall

$$\Phi = \frac{1}{\sqrt{2}}(v+H)\, e^{i\phi/v} = \frac{1}{\sqrt{2}}\left(\frac{\mu}{\sqrt{\lambda}} + H + i\phi + \dots\right), \tag{4.19}$$

where the dots stand for nonlinear terms in ϕ. Next we make a gauge transformation (see eq. (1.2))

$$\Phi \to \Phi' = e^{-i\phi/v}\,\Phi. \tag{4.20}$$

In other words, we fix the gauge such that the imaginary part of Φ vanishes. Under the gauge transformation eq. (4.20) the gauge field transforms according to (see eq. (1.11))

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{gv}[\partial_\mu \phi]. \tag{4.21}$$

It is in fact the superposition of A_μ and ϕ which make up the physical field. Note that the change from A_μ to A'_μ made in eq. (4.21) affects only the longitudinal component. If we now express the Lagrangian in terms of Φ' and A'_μ there will be no mixing term. Even better, the ϕ field vanishes altogether! This can easily be seen by noting that under a gauge transformation the covariant derivative $D_\mu\Phi$ transforms in the same way as Φ, thus

$$D_\mu\Phi \to (D_\mu\Phi)' = e^{-i\phi/v} D_\mu\Phi = e^{-i\phi/v}\frac{1}{\sqrt{2}}\left(\partial_\mu H + igA'_\mu(v+H)\right), \tag{4.22}$$

and $(D_\mu\Phi)'^*(D^\mu\Phi)'$ is independent of ϕ. Performing the algebra (and dropping the $'$ for the A-field) we get the Lagrangian in the unitary gauge

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu H\partial^\mu H + \frac{M_A^2}{2}A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{M_H^2}{2}H^2 \\ &+ gM_A A_\mu A^\mu H + \frac{g^2}{2}A_\mu A^\mu H^2 - \frac{\lambda}{4}H^4 - \sqrt{\frac{\lambda}{2}}M_H H^3,\end{aligned} \tag{4.23}$$

with M_A and M_H as defined in eqs. (4.16) and (4.14), respectively. All the terms quadratic in A_μ may be written (in momentum space) as

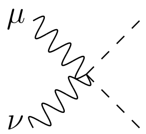
$$A_\mu(-p)\left(-g^{\mu\nu}p^2 + p^\mu p^\nu + g^{\mu\nu}M_A^2\right)A_\nu(p). \tag{4.24}$$

The gauge boson propagator is the inverse of the coefficient of $A_\mu(-p)A_\nu(p)$, which is

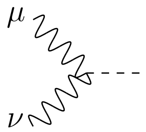
$$-i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{M_A^2}\right)\frac{1}{(p^2 - M_A^2)}. \tag{4.25}$$

This is the usual expression for the propagator of a massive spin-one particle, eq. (4.2). The only other remaining particle is the scalar, H, with mass $m_H = \sqrt{2}\,\mu$, which is the

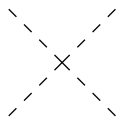
Higgs boson. This is a physical particle, which interacts with the gauge boson and also has cubic and quartic self-interactions. The Lagrangian given in eq. (4.23) leads to the following vertices and Feynman rules:



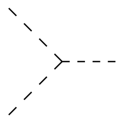
$2\,i\,e^2 g_{\mu\nu}$



$2\,i\,e M_A g_{\mu\nu}$



$6\,i\,\lambda$



$6\,i\,m_H\sqrt{2\lambda}$

The advantage of the unitary gauge is that no unphysical particles appear, i.e. the ϕ-field has completely disappeared. The disadvantage is that the propagator of the gauge field, eq. (4.25), behaves as p^0 for $p \to \infty$. As discussed in section 4.1 this seems to indicate that the theory is non-renormalizable. It seems that we have not gained anything at all by breaking the theory spontaneously rather than by simply adding a mass term by hand. Fortunately this is not true. In order to see that the theory is still renormalizable, in spite of eq. (4.25), it is very useful to consider a different type of gauges, namely the R_ξ gauges discussed in the next subsection.

4.6 R_ξ Gauges (Feynman Gauge)

The class of R_ξ gauges is a more conventional way to fix the gauge. Recall that in QED we fixed the gauge by adding a term, eq. (1.21), in the Lagrangian. This is exactly what we do here. The gauge fixing term we are adding to the Lagrangian density eq. (4.6) is

$$\begin{aligned}\mathcal{L}_R &\equiv -\frac{1}{2(1-\xi)}\left(\partial_\mu A^\mu - (1-\xi)M_A\phi\right)^2 \\ &= -\frac{1}{2(1-\xi)}\partial_\mu A^\mu \partial_\nu A^\nu + M_A\phi\partial_\mu A^\mu - \frac{1-\xi}{2}M_A^2\phi^2. \end{aligned} \tag{4.26}$$

Again, the special value $\xi = 0$ corresponds to the Feynman gauge. The second term in eq. (4.26) cancels precisely the mixing term in eq. (4.15). Thus, we have achieved our goal. Note however, that in this case, contrary to the unitary gauge, the unphysical ϕ-field does not disappear. The first term in eq. (4.26) is bilinear in the gauge field, thus it contributes to the gauge-boson propagator. The terms bilinear in the A-field are

$$-\frac{1}{2}A^{\mu}(-p)\left(-g_{\mu\nu}(p^2 - M_a^2) + p_\mu p_\nu - \frac{p_\mu p_\nu}{1-\xi}\right)A^{\nu}(p) \tag{4.27}$$

which leads to the gauge boson propagator

$$\frac{-i}{(p^2 - M_A^2)}\left(g_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2 - (1-\xi)M_A^2}\right). \tag{4.28}$$

In the Feynman gauge, the propagator becomes particularly simple. The crucial feature of eq. (4.28), however, is that this propagator behaves as p^{-2} for $p \to \infty$. Thus, this class of gauges is manifestly renormalizable. There is, however, a price to pay: The Goldstone boson is still present. It has acquired a mass, M_A, from the gauge fixing term, and it has interactions with the gauge boson, with the Higgs scalar and with itself. Furthermore, for the purposes of higher order corrections in non-Abelain theories, we need to introduce Faddeev-Popov ghosts which interact with the gauge bosons, the Higgs scalar and the Goldstone bosons.

Let us stress that there is no contradiction at all between the apparent non-renormalizability of the theory in the unitary gauge and the manifest renormalizability in the R_ξ gauge. Since physical quantities are gauge invariant, any physical quantity can be calculated in a gauge where renormalizability is manifest. As mentioned above, the price we pay for this is that there are more particles and many more interactions, leading to a plethora of Feynman diagrams. We therefore only work in such gauges if we want to compute higher order corrections. For the rest of these lectures we shall confine ourselves to tree-level calculations and work solely in the unitary gauge.

Nevertheless, one cannot over-stress the fact that it is only when the gauge bosons acquire masses through the Higgs mechanism that we have a renormalizable theory. It is this mechanism that makes it possible to write down a consistent Quantum Field Theory which describes the weak interactions.

4.7 Summary

- In the case of a gauge theory the Goldstone bosons provide the longitudinal component of the gauge bosons, which therefore acquire a mass. The mass is proportional to the magnitude of the vacuum expectation value and the gauge coupling constant. The Goldstone bosons themselves are unphysical.

- It is possible to work in the unitary gauge where the Goldstone boson fields are set to zero.
- When gauge bosons acquire masses by this (Higgs) mechanism, renormalizability is maintained. This can be seen explicitly if one works in a R_ξ gauge, in which the gauge boson propagator decreases like $1/p^2$ as $p \to \infty$. This is a necessary condition for renormalizability. If one does work in such a gauge, however, one needs to work with Goldstone boson fields, even though the Goldstone bosons are unphysical. The number of interactions and the number of Feynman graphs required for the calculation of some processes is then greatly increased.

5 The Standard Model with one Family

To write down the Lagrangian of a theory, one first needs to choose the symmetries (gauge and global) and the particle content, and then write down every allowed renormalizable interaction. In this section we shall use this recipe to construct the Standard Model with one family. The Lagrangian should contain pieces

$$\mathcal{L}_{(SM,1)} = \mathcal{L}_{\text{gauge bosons}} + \mathcal{L}_{\text{fermion masses}} + \mathcal{L}_{\text{fermionKT}} + \mathcal{L}_{\text{Higgs}}. \tag{5.1}$$

The terms are written out in eqns. (5.15), (5.29), (5.30) and (5.55).

5.1 Left- and Right- Handed Fermions

The weak interactions are known to violate parity. Parity non-invariant interactions for fermions can be constructed by giving different interactions to the "left-handed" and "right-handed" components defined in eq. (5.4). Thus, in writing down the Standard Model, we will treat the left-handed and right-handed parts separately.

A Dirac field, ψ, representing a fermion, can be expressed as the sum of a left-handed part, ψ_L, and a right-handed part, ψ_R,

$$\psi \;=\; \psi_L \;+\; \psi_R, \tag{5.2}$$

where

$$\psi_L \;=\; P_L\,\psi \quad \text{with} \quad P_L = \frac{(1-\gamma_5)}{2}, \tag{5.3}$$

$$\psi_R \;=\; P_R\,\psi \quad \text{with} \quad P_R = \frac{(1+\gamma_5)}{2}. \tag{5.4}$$

P_L and P_R are projection operators, i.e.

$$P_L\,P_L \;=\; P_L, \quad P_R\,P_R \;=\; P_R \quad \text{and} \quad P_L\,P_R \;=\; 0 \;=\; P_R\,P_L. \tag{5.5}$$

They project out the left-handed (negative) and right-handed (positive) *chirality* states of the fermion, respectively. This is the definition of chirality, which is a property of fermion fields, but not a physical observable.

The kinetic term of the Dirac Lagrangian and the interaction term of a fermion with a vector field can also be written as a sum of two terms, each involving only one chirality

$$\overline{\psi}\,\gamma^\mu\partial_\mu\psi \;=\; \overline{\psi_L}\,\gamma^\mu\partial_\mu\psi_L \;+\overline{\psi_R}\,\gamma^\mu\partial_\mu\psi_R, \tag{5.6}$$

$$\overline{\psi}\,\gamma^\mu A_\mu\psi \;=\; \overline{\psi_L}\,\gamma^\mu A_\mu\psi_L \;+\overline{\psi_R}\,\gamma^\mu A_\mu\psi_R. \tag{5.7}$$

On the other hand, a mass term mixes the two chiralities:

$$m\overline{\psi}\,\psi \;=\; m\overline{\psi_L}\,\psi_R \;+ m\overline{\psi_R}\,\psi_L. \tag{5.8}$$

Exercise 5.1

Use $(\gamma_5)^2 = 1$ to verify eq. (5.5) and $\overline{\psi} = \psi^\dagger\gamma^0$, $\gamma^{5\,\dagger} = \gamma^5$ as well as $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ to verify eq. (5.7).

In the limit where the fermions are massless (or sufficiently relativistic), chirality becomes *helicity*, which is the projection of the spin on the direction of motion and which is a physical observable. Thus, if the fermions are massless, we can treat the left-handed and right-handed chiralities as separate particles of conserved helicity. We can understand this physically from the following simple consideration. If a fermion is massive and is moving in the *positive* z direction, along which its spin is having a *positive* component so that the helicity is *positive* in this frame, one can always boost into a frame in which the fermion is moving in the *negative* z direction, but with this spin component unchanged. In the new frame the helicity will hence be *negative*. On the other hand, if the particle is massless and travels with the speed of light, no such boost is possible, and in that case helicity/chirality is a good quantum number.

Exercise 5.2

For a massless spinor

$$u(p) \;=\; \frac{1}{\sqrt{E}}\begin{pmatrix} E\,\chi \\ \vec{\sigma}\cdot\vec{p}\,\chi \end{pmatrix},$$

where χ is a two-component spinor, show that

$$(1 \pm \gamma^5)u(p)$$

are eigenstates of $\vec{\sigma}\cdot\vec{p}/E$ with eigenvalues ± 1, respectively. Take

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and in 4×4 matrix notation $\vec{\sigma}\cdot\vec{p}$ means

$$\begin{pmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{pmatrix}.$$

5.2 Symmetries and Particle Content

We have made all the preparations to write down a gauge invariant Lagrangian. We now only have to pick the gauge group and the matter content of the theory. It should be noticed that there are no theoretical reasons to pick a certain group or certain matter content. To match experimental observations we pick the gauge group for the Standard Model to be

$$U(1)_Y \times SU(2) \times SU(3). \tag{5.9}$$

To indicate that the abelian $U(1)$ group is *not* the gauge group of QED but of hypercharge a subscript Y has been added. The corresponding coupling and gauge boson is denoted by g' and B^μ respectively.

The $SU(2)$ group has three generators ($\mathbf{T}_a = \sigma_a/2$), the coupling is denoted by g and the three gauge bosons are denoted by $W^1_\mu, W^2_\mu, W^3_\mu$. None of these gauge bosons (and neither B_μ) are physical particles. As we will see, linear combinations of these gauge bosons will make up the photon as well as the $W^\pm$ and the Z bosons.

Finally, the $SU(3)$ is the group of the strong interaction. The corresponding eight gauge bosons are the gluons. In this section we will concentrate on the other two groups, with one generation of fermions. The strong interaction is dealt with in section 3, and extra generations are introduced in the next chapter.

As matter content for the first family, we have

$$q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix};\ u_R;\ d_R;\ \ell_L \equiv \begin{pmatrix} \nu_L \\ e_L \end{pmatrix};\ e_R;\ \{\nu_R!!\}. \tag{5.10}$$

Note that a right-handed neutrino ν_R has appeared. It is a gauge singlet (no strong interaction, no weak interactions, no electric charge), so is unneccessary in a model with massless neutrinos. However, neutrinos are now known to have small masses, which can be described by adding the right-handed field ν_R. Neutrino masses will be discussed further in chapter 7.

Note also that the left- and right-handed fermion components have been given different weak interactions. The Standard Model is constructed this way, because the weak interactions are known to violate parity. The left-handed components form doublets under $SU(2)$ whereas the right-handed components are singlets. This means that under $SU(2)$ gauge transformations we have

$$e_R \rightarrow e'_R = e_R, \tag{5.11}$$

$$\ell_L \rightarrow \ell'_L = e^{-i\omega^a \mathbf{T}^a} \ell_L. \tag{5.12}$$

Thus, the $SU(2)$ singlets e_R, ν_R, u_R and d_R are invariant under $SU(2)$ transformations and do not couple to the corresponding gauge bosons $W_\mu^1, W_\mu^2, W_\mu^3$.

Since this separation of the electron into its left- and right-handed helicity only makes sense for a massless electron we also need to assume that the electron *is* massless in the exact $SU(2)$ limit and that the mass for the electron arises as a result of spontaneous symmetry breaking in a similar way as the masses for the gauge bosons arise. We will come back to this later.

Under $U(1)_Y$ gauge transformations the matter fields transform as

$$\psi \to \psi' = e^{-i\omega Y(\psi)}\psi \tag{5.13}$$

where Y is the hypercharge of the particle under consideration. It is chosen to give the observed electric charge of the particles. The explicit values for the hypercharges of the particles listed in eq. (5.10) are as follows:

$$Y(\ell_L) = -\frac{1}{2},\ Y(e_R) = -1,\ Y(\nu_R) = 0,\ Y(q_L) = \frac{1}{6},\ Y(u_R) = \frac{2}{3},\ Y(d_R) = -\frac{1}{3}. \tag{5.14}$$

Under $SU(3)$ the lepton fields ℓ_L, e_R, ν_R are singlets, i.e. they do not transform at all. This means that they do not couple to the gluons. The quarks on the other hand form triplets under $SU(3)$. The strong interaction does not distinguish between left- and right-handed particles.

We have now listed all fermions that belong to the first family, together with their transformation properties under the various gauge transformations. However, since we ultimately want massive weak gauge bosons, we will have to break the $U(1)_Y \times SU(2)$ gauge group spontaneously, by introducing some type of Higgs scalar. The transformation properties of this scalar will be deduced in the discussion of fermion masses.

5.3 Kinetic Terms for the Gauge Bosons

The gauge kinetic terms for abelian and non-abelian theories were presented in the first two lectures. From the general expression of eq. (2.21), we extract for the SM gauge bosons:

$$\mathcal{L} \;=\; -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} - \frac{1}{4}F^A_{\mu\nu}F^{A\,\mu\nu} + \mathcal{L}_{\text{gauge}-\text{fixing}} + \mathcal{L}_{\text{FP ghosts}}. \tag{5.15}$$

Here $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ is the hypercharge field strength, the second term contains the $SU(2)$ field strength, so a runs from one to three (over the three vector bosons of $SU(2)$), and the third term is the gluon kinetic term, so $A = 1\ldots 8$. To do an explicit perturbative calculation, additional gauge fixing terms, and Fadeev-Popov ghosts, must be included. The form of these terms depends on the choice of gauge.

5.4 Fermion Masses and Yukawa Couplings

We cannot have an explicit mass term for the quarks or electrons, since a mass term mixes left-handed and right-handed fermions and we have assigned these to different multiplets of weak SU(2). However, if an SU(2) doublet Higgs is introduced, there is a gauge invariant interaction that will look like a mass when the Higgs gets a vacuum expectation value ("vev"). Such an interaction is called a 'Yukawa interaction' and is written as

$$\mathcal{L}_{\text{Yukawa}} \;=\; -\,Y_e\,\overline{l_L}^{\,i}\,\Phi_i e_R \;+\; \text{h.c.}\,, \tag{5.16}$$

where h.c. means 'hermitian conjugate'. Note that the Higgs doublet must have $Y = 1/2$ to ensure that this term has zero weak hypercharge.

Recalling eq. (5.19) we introduce a scalar "Higgs" field, which is a doublet under $SU(2)$, singlet under $SU(3)$ (no colour), and has a scalar potential as given in eq. (4.9), i.e.

$$V(\Phi) \;=\; -\mu^2\Phi^*\Phi + \lambda\,|\Phi^*\Phi|^2\,. \tag{5.17}$$

This potential has a minimum at $\Phi^*\Phi = \frac{1}{2}\mu^2/\lambda$, so some component of the Higgs doublet should get a vev. In the unitary gauge, this vev can be written as

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{5.18}$$

with $v = \mu/\sqrt{\lambda}$.

Recall from the previous chapter that Φ can be written as its "radial" degree of freedom times an exponential containing the broken generators of the gauge symmetry:

$$\Phi \;=\; \frac{e^{i(\omega_a \mathbf{T}^a - \omega_3 \mathbf{Y})}}{\sqrt{2}}\begin{pmatrix} 0 \\ v\,+\,H \end{pmatrix}. \tag{5.19}$$

The unitary gauge choice consists of absorbing this exponential with a gauge transformation, so that in the unitary gauge eq. (5.16) is

$$\mathcal{L}_{\text{Yukawa}} \;=-\,\frac{Y_e}{\sqrt{2}}\begin{pmatrix} \overline{\nu_L} & \overline{e_L} \end{pmatrix}\begin{pmatrix} 0 \\ v\,+\,H \end{pmatrix} e_R \;+\; \text{h.c.}\,. \tag{5.20}$$

The part proportional to the vev is simply

$$-\,\frac{Y_e\,v}{\sqrt{2}}\left(\overline{e_L}\,e_R + \overline{e_R}\,e_L\right) \;=\; \frac{Y_e\,v}{\sqrt{2}}\,\overline{e}\,e\,, \tag{5.21}$$

and we see that the electron has acquired a mass which is proportional to the vev of the scalar field. This immediately gives us a relation for the Yukawa coupling in terms of the electron mass, m_e, and the W mass, M_W:

$$Y_e \;=\; g\,\frac{m_e}{\sqrt{2}\,M_W}. \tag{5.22}$$

Thus, as for the gauge bosons, the strength of the coupling of the Higgs to fermions is proportional to the mass of the fermions.

The quarks also acquire a mass through the spontaneous symmetry breaking mechanism, via their Yukawa coupling with the scalars. The interaction term

$$-\,Y_d\,\overline{q_L}^{\,i}\,\Phi_i\,d_R \;+\; \text{h.c.} \tag{5.23}$$

gives a mass to the d quark when we replace Φ_i by its vev. This mass m_d is given by

$$m_d \;=\; \frac{Y_d}{\sqrt{2}}\,v \;=\; \sqrt{2}\,\frac{Y_d\,M_W}{g}\,. \tag{5.24}$$

Since the vev is in the lower component of the Higgs doublet, we must do a little more work to obtain a mass for the upper element u of the quark doublet. In the case of $SU(2)$ there is a second way in which we can construct an invariant for the Yukawa interaction:

$$-\,Y_u\;\epsilon_{ij}\,\overline{q_L}^{\,i}\,\Phi^{*\,j}u_R +\; \text{h.c.} \qquad (i,j \;=\; 1,2), \tag{5.25}$$

where ϵ_{ij} is the two-dimensional antisymmetric tensor. Note that

$$\Phi^c = \epsilon_{ij}\Phi^{j*} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Phi_+^* \\ \Phi_0^* \end{pmatrix} \tag{5.26}$$

has $Y = -1/2$, as required by the $U(1)$ symmetry. This term does indeed give a mass m_u to the u quark, where

$$m_u \;=\; \frac{Y_u}{\sqrt{2}}\,v \;=\; \sqrt{2}\,\frac{Y_u\,M_W}{g}\,. \tag{5.27}$$

So the SM Higgs scalar couples to both the u and d quark, with interaction terms

$$-\,g\frac{m_u}{2\,M_W}\,\overline{u}\,H\,u \;-\; g\frac{m_d}{2\,M_W}\,\overline{d}\,H\,d\,. \tag{5.28}$$

The terms in the Lagrangian that give masses to the first generation quarks and charged leptons are

$$\mathcal{L}_{\text{fermion masses}} \;=\; -\,Y_e\,\overline{l_L}^{\,i}\,\Phi_i e_R \;-\; Y_d\,\overline{q_L}^{\,i}\,\Phi_i\,d_R \;-\; Y_u\;\epsilon_{ij}\,\overline{q_L}^{\,i}\,\Phi^{*\,j}u_R +\; \text{h.c.}\,. \tag{5.29}$$

We could also have included a Yukawa mass term for the neutrinos: $-\,Y_\nu\;\epsilon_{ij}\,\overline{\ell_L}^{\,i}\,\Phi^{*\,j}\nu_R +$ h.c. However, neutrino masses do not neccessarily arise from a Yukawa interaction (this will be discussed in chapter 7).

5.5 Kinetic Terms for Fermions

The fermionic kinetic terms should be familiar from chapter 2:

$$\begin{aligned}\mathcal{L}_{\text{fermionKT}} &= i\,\overline{\ell_L}^T\,\gamma^\mu \mathbf{D}_\mu\,\ell_L + i\,\overline{e_R}\,\gamma^\mu D_\mu\,e_R + i\,\overline{\nu_R}\,\gamma^\mu \partial_\mu\,\nu_R \\ &\quad + i\,\overline{q_L}^T\,\gamma^\mu \mathbf{D}_\mu\,q_L + i\,\overline{d_R}\,\gamma^\mu \mathbf{D}_\mu\,d_R + i\,\overline{u_R}\,\gamma^\mu \partial_\mu\,u_R \end{aligned} \tag{5.30}$$

where the covariant derivatives include the hypercharge, $SU(2)$ and $SU(3)$ gauge bosons as required. For instance:

$$\mathbf{D}_\mu = \partial_\mu + ig\mathbf{T}^a W_\mu^a + ig'Y(\ell_L)B_\mu \qquad \text{for } \ell_L\,, \tag{5.31}$$

$$D_\mu = \partial_\mu + ig'Y(e_R)B_\mu \qquad \text{for } e_R\,, \tag{5.32}$$

$$\mathbf{D}_\mu = \partial_\mu + ig_s\mathbf{T_s}^a G_\mu^a + ig'Y(d_R)B_\mu \qquad \text{for } d_R\,, \tag{5.33}$$

where the strong coupling (g_s), the eight generators of $SU(3)$ ($\mathbf{T_s}^a$) and the corresponding gluon fields (G_μ^a) have been introduced, and $Y(f)$ is the hypercharge of fermion f.

This gives the following interaction terms between the leptons and the gauge bosons:

$$-\frac{g}{2}\begin{pmatrix}\overline{\nu_L}\\ \overline{e_L}\end{pmatrix}^T \gamma^\mu \left(\begin{pmatrix} W_\mu^0 & \sqrt{2}\,W_\mu^- \\ \sqrt{2}\,W_\mu^+ & -W_\mu^0\end{pmatrix} - \tan\theta_W B_\mu\right)\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} - i\,g\,\tan\theta_W\,\overline{e_R}\,\gamma^\mu B_\mu e_R\,, \tag{5.34}$$

where we have used

$$Z_\mu \equiv \cos\theta_W\,W_\mu^3 - \sin\theta_W B_\mu\,, \tag{5.35}$$

$$A_\mu \equiv \cos\theta_W\,B_\mu + \sin\theta_W W_\mu^3 \tag{5.36}$$

to replace B_μ and W_μ^3 by the physical particles Z_μ and A_μ. (In the exercises of chapter 2 these definitions followed from requiring that the photon does not interact with the neutrino. In section 5.6 we will see that the photon is also massless).

Writing out the projection operators for left- and right-handed fermions, eqs. (5.3) and (5.4), we obtain the following interactions:

1. A coupling of the charged vector bosons $W^\pm$ which mediate transitions between neutrinos and electrons (or u and d quarks) with an interaction term

$$-\frac{g}{2\sqrt{2}}\,\overline{\nu}\,\gamma^\mu\left(1-\gamma^5\right)e\,W_\mu^- - \frac{g}{2\sqrt{2}}\,\overline{u}\,\gamma^\mu\left(1-\gamma^5\right)d\,W_\mu^- + \text{h.c.} \tag{5.37}$$

(h.c. means 'hermitian conjugate' and gives the interaction involving an emitted W_μ^+ where the incoming particle is a neutrino (or u) and the outgoing particle is an electron (or d).)

2. The usual coupling of the photon with the charged fermions is (using, for instance, the relation eq. (5.54)):

$$g\,\sin\theta_W\,\overline{e}\,\gamma^\mu\,e\,A_\mu - \frac{2}{3}\,g\,\sin\theta_W\,\overline{u}\,\gamma^\mu\,u\,A_\mu\;+\;\frac{1}{3}\,g\,\sin\theta_W\,\overline{d}\,\gamma^\mu\,d\,A_\mu\,. \tag{5.38}$$

Note that the left- and right-handed fermions have exactly the same coupling to the photon so that the electromagnetic coupling turns out to be purely vector (i.e. no γ^5 term).

3. The coupling of neutrinos to the neutral weak gauge boson Z_μ:

$$-\frac{g}{4\cos\theta_W}\,\overline{\nu}\,\gamma^\mu\left(1-\gamma^5\right)\,\nu\,Z_\mu\,. \tag{5.39}$$

4. The coupling of both the left- and right-handed electron to the Z:

$$\frac{g}{4\cos\theta_W}\,\overline{e}\,\left(\gamma^\mu\left(1-\gamma^5\right)-4\sin^2\theta_W\gamma^\mu\right)\,e\,Z_\mu\,. \tag{5.40}$$

5. The coupling of the quarks to the Z can be written in the general form

$$-\frac{g}{2\cos\theta_W}\,\overline{q_i}\,\left(T_i^3\,\gamma^\mu\left(1-\gamma^5\right)-\;2Q_i\sin^2\theta_W\gamma^\mu\right)\,q_i\,Z_\mu\,, \tag{5.41}$$

where quark i has the third component of weak isospin T_i^3 and electric charge Q_i.

From these terms in the Lagrangian we can directly read off the Feynman rules for the three-point vertices with two fermions and one weak gauge boson. Then we can use these vertices to calculate weak interactions of the quarks and leptons. This allows us, for example, to calculate the total decay width of the Z or W boson, by calculating the decay width into all possible quarks and leptons. However, quarks are not free particles, so for exclusive processes, in which we trigger on known initial or final state hadrons, information is needed about the probability to find a quark with given properties inside an initial hadron or the probability that a quark with given properties will decay ("fragment") into a final state hadron.

Exercise 5.3

The decay rate for the Z into a fermion-antifermion pair, $Z \to f\bar{f}$, is

$$\Gamma = \frac{1}{2M_Z} \int d^{\mathrm{LIPS}} \, |\mathcal{M}|^2 = \frac{1}{64\,\pi^2\, M_Z} \int d\Omega \, |\mathcal{M}|^2,$$

where d^{LIPS} stands for the Lorentz invariant phase space measure for the two final-state fermions, and $\int d\Omega$ is the integral over the solid angle (of one final-state particle).

Write the general interaction term for the coupling of the Z boson to a fermion as

$$-\frac{g}{2\cos\theta_W} \gamma^\mu \left(v_f - a_f \, \gamma^5 \right).$$

Show that the squared matrix element, summed over the spins of the (outgoing) fermions and averaged over the spin of the (incoming) Z boson is

$$|\mathcal{M}|^2 = -\frac{1}{12} \, g_{\mu\nu} \, \frac{g^2}{\cos^2\theta_W} \left((v_f)^2 + (a_f)^2 \right) \mathrm{Tr} \left(\gamma^\mu \gamma \cdot k_1 \gamma^\nu \gamma \cdot k_2 \right),$$

where k_1 and k_2 are the momenta of the outgoing fermions and the gauge polarization sum is

$$\sum_\lambda \epsilon_\mu^{(\lambda)*} \, \epsilon_\nu^{(\lambda)} = -g_{\mu\nu} + \frac{q_\mu \, q_\nu}{M_Z^2}$$

($q = k_1 + k_2$ is the initial momentum of the Z boson). Hence show that

$$\Gamma = \frac{1}{48\,\pi} \frac{g^2}{\cos^2\theta_W} \left((v_f)^2 + (a_f)^2 \right) M_Z.$$

Neglect the masses of the fermions in comparison to the Z mass.

Exercise 5.4

The Z boson can decay leptonically into a pair of neutrinos or charged leptons of all three generations and hadronically into u quarks, d quarks, c quarks, s quarks, or b quarks (c quarks couple like u quarks, whereas s quarks and b quarks couple like d quarks). Deduce the values of v_f and a_f for each of these cases and consequently estimate the decay width of the Z boson. (The current experimental value is 2.4952 ± 0.0023 GeV.)

[Take $M_Z = 91.19$ GeV, $\sin^2\theta_W = 0.23$, and the fine-structure constant $\alpha = 1/129$ (why this value?)].

5.6 The Higgs Part and Gauge Boson Masses

The Higgs doublet Lagrangian should contain a "spontaneous symmetry breaking" potential which will give the Higgs a vev and self-interactions, and kinetic terms which will generate the gauge boson masses and interactions between the Higgs and the gauge bosons. We first consider the potential:

$$V(\Phi) \;=\; -\mu^2 \Phi_i^* \Phi^i + \lambda \left(\Phi_i^* \Phi^i\right)^2 . \tag{5.42}$$

This potential has a minimum at $\Phi_i^+ \Phi_i = \frac{1}{2}\mu^2/\lambda$. Writing Φ in the form of eq. (5.19) and replacing this in the potential eq. (5.42), we find that we get a mass term for the real Higgs field H, with value $m_H = \sqrt{2}\mu$. As expected, the ω_a do not appear in the potential. In an ungauged theory, they would be the massless goldstone bosons. In a gauge theory like the Standard Model, they will reappear as the longitudinal degrees of freedom of the massive gauge bosons.

The remaining term of the Φ Lagrangian is the kinetic term $(D_\mu \Phi)^\dagger (D^\mu \Phi)$. Looking at this term more carefully will help us to understand where the "physical" gauge bosons (i.e. the $W^\pm$, Z and photon) come from, and how they are related to the $W_\mu^1, W_\mu^2, W_\mu^3, B_\mu$. To see the effect of the Higgs vev on the gauge boson masses, it is most simple to work in the unitary gauge, that is, we absorb the exponential of eq. (5.19) with a gauge transformation. In this gauge, the covariant derivative acting on the Higgs doublet is

$$\mathbf{D}_\mu \Phi \;=\; \frac{1}{\sqrt{2}} \left(\partial_\mu \; + i\,\frac{g}{2} \begin{pmatrix} W_\mu^3 & \sqrt{2}\,W_\mu^- \\ \sqrt{2}\,W_\mu^+ & -W_\mu^3 \end{pmatrix} + i\,\frac{g'}{2} B_\mu \right) \begin{pmatrix} 0 \\ v \;+\; H \end{pmatrix} , \tag{5.43}$$

so that

$$|\mathbf{D}_\mu \Phi|^2 = \frac{1}{2} (\partial_\mu H)^2 + \frac{g^2 v^2}{4} W^{+\,\mu} W_\mu^- + \frac{v^2}{8} \left(g\, W_\mu^3 - g' B_\mu \right)^2 \;\; + \text{interaction terms}\,, \tag{5.44}$$

where the 'interaction terms' are terms involving three fields (two gauge fields and the H-field). Eq. (5.44) tells us that the W_μ^3 and B_μ fields mix (as do W_μ^1 and W_μ^2) and the physical gauge bosons must be superpositions of these fields, such that there are no mixing terms. Thus we define

$$Z_\mu \;\equiv\; \cos\theta_W \, W_\mu^3 - \sin\theta_W B_\mu \,, \tag{5.45}$$

$$A_\mu \;\equiv\; \cos\theta_W \, B_\mu + \sin\theta_W W_\mu^3 \,, \tag{5.46}$$

with the weak mixing angle θ_W ("Weinberg angle") defined by

$$\tan\theta_W \equiv \frac{g'}{g} \,. \tag{5.47}$$

With this eq. (5.44) is rewritten as

$$|\mathbf{D}_\mu \Phi|^2 \;=\; \frac{1}{2}(\partial_\mu H)^2 + \frac{g^2v^2}{4} W^+_\mu W^{-\,\mu} + \frac{v^2g^2}{8\cos^2\theta_W} Z_\mu Z^\mu + 0 A_\mu A^\mu \,. \qquad (5.48)$$

Here we see how $SU(2)$ and $U(1)$ are unified (or at least 'entangled') in the sense that the neutral gauge boson that acquires a mass through the Higgs mechanism is the linear superposition of a gauge boson from the $SU(2)$ and the $U(1)_Y$ gauge boson.

From eq. (5.48) we can read off the masses of the gauge bosons. The last term tells us that the linear combination eq. (5.46) remains massless. This field is identified with the photon. For the other fields we have

$$M_W = \frac{1}{2} g v \,, \qquad M_Z = \frac{1}{2}\frac{gv}{\cos\theta_W} \,. \qquad (5.49)$$

The Z boson mediates the neutral current weak interactions. These were not observed until after the development of the theory. From the magnitude of amplitudes involving weak neutral currents (exchange of a Z boson), one can infer the (tree level) magnitude of the weak mixing angle, θ_W. The ratio of the masses of the Z and W bosons is a prediction of the Standard Model. More precisely, we define a quantity known as the ρ-parameter by

$$M_W^2 = \rho\, M_Z^2 \cos^2\theta_W \,. \qquad (5.50)$$

In the Standard Model $\rho = 1$ at tree level. In higher orders there is a small correction, which depends on the definition used for $\sin\theta_W$ (that is, which loop corrections are included in $\sin\theta_W$). Note that the ρ-parameter would be very different from one if the symmetry breaking were due to a scalar multiplet which was not a doublet of weak isospin. Accurate measurements of the ρ-parameter and other so-called electro-weak precision observables, together with their prediction at higher order within the SM, serve as very powerful tests of the SM. The Higgs enters in virtual loops, allowing for an indirect determination of its mass through fits of the predictions to the data (see the homepage of the Electroweak Working Group, `http://lepewwg.web.cern.ch/LEPEWWG` for more information).

The spontaneous symmetry breaking mechanism breaks $SU(2) \times U(1)_Y$ down to $U(1)$. It is this surviving $U(1)$ that is identified as the $U(1)$ of electromagnetism. It is not the $U(1)_Y$ of the original gauge group but a set of transformations generated by a particular linear combination of the original $U(1)$ and rotations about the third axis of weak isospin. To see this we note that the explicit representation of the generator $\mathbf{Y}$ as a 2×2 matrix, which can be combined with the explicit representation of $\mathbf{T}^1$, $\mathbf{T}^2$ and $\mathbf{T}^3$, is given by

$$Y = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,. \qquad (5.51)$$

The factor 1/2 ensures the normalization[12] condition eq. (2.7). Using eq. (5.51) it can easily be seen that the symmetry associated with the generator

$$Q \equiv Y + T^3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.52}$$

is not broken, i.e. $Q|0\rangle = 0$ (see eq. (4.5)). Thus, starting with four generators, we get only three Goldstone bosons. These will become the longitudinal components of three gauge bosons, thereby giving them a mass, whereas the fourth is left massless.

The coupling of any particle to the photon is always proportional to

$$g \sin\theta_W \,(Y \;+\; T_3) \;=\; g \sin\theta_W \, Q \,. \tag{5.53}$$

Thus we can identify $g \sin\theta_W$ with one unit of electric charge, and we have the relationship between the weak coupling g and the electron charge e,

$$e \;=\; g \sin\theta_W \,. \tag{5.54}$$

We end this subsection by giving the remaining pieces of the SM Lagrangian from eqs. (5.44) and (5.42),

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} &= |\mathbf{D}_\mu \Phi|^2 \; - \mu^2 \Phi_i^* \Phi^i + \lambda \left(\Phi_i^* \Phi^i\right)^2 \\ &= \frac{1}{2}(\partial_\mu H)^2 + \mu^2 H^2 + \frac{g^2 v^2}{4} W^{+\mu} W^-_\mu + \frac{v^2 g^2}{8\cos^2\theta_W} Z_\mu Z^\mu \\ &\quad + \text{interaction terms} \,. \end{aligned} \tag{5.55}$$

5.7 Classifying the Free Parameters

The free parameters in the Standard Model for one generation are:

- The two gauge couplings for the $SU(2)$ and $U(1)$ gauge groups, g and g'.
- The two parameters μ and λ in the scalar potential $V(\Phi)$.
- The Yukawa coupling constants Y_u, Y_d, Y_e and Y_ν.

It is convenient to replace these parameters by others, which are more directly measurable in experiments, namely $e, \sin\theta_W, m_e$ and m_W, and m_H, m_u, m_d and m_ν. (Note that the gauge

[12] We warn the reader that in the literature sometimes a different normalization is used such that eq. (5.52) reads $Q = Y/2 + T^3$.

sector is well measured, but the quark masses are not directly observable; we have yet to find the Higgs, and although we see neutrino mass differences, measuring the absolute mass scale is difficult — and the neutrino masses might not be directly proportional to Yukawa couplings anyway.) The relation between these physical parameters and the parameters of the initial Lagrangian are

$$\tan\theta_W = \frac{g'}{g}, \quad (5.56)$$

$$e = g\sin\theta_W, \quad (5.57)$$

$$m_H = \sqrt{2}\mu, \quad (5.58)$$

$$M_W = \frac{g\,\mu}{2\sqrt{\lambda}}, \quad (5.59)$$

$$m_e = Y_e\frac{\mu}{\sqrt{\lambda}}. \quad (5.60)$$

Note that when we add more generations of fermions, we will acquire more parameters: additional masses (or yukawa couplings, i.e. 4 parameters per generation), and also mixing angles, as we will see in the next chapter.

In terms of these measured quantities, the Z mass, M_Z, and the Fermi-coupling, G_F, are *predictions* of the SM (although historically G_F was known for many years before the discovery of the W boson, and its value was used to predict the W mass).

5.8 Summary

- Weak interactions are mediated by the $SU(2)$ gauge bosons, which act only on the left-handed components of fermions.
- The (left-handed) neutrino and left-handed component of the electron form an $SU(2)$ doublet, whereas the right-handed components of the electron and neutino are $SU(2)$ singlets. Similarly for the quarks.
- There is also a weak hypercharge $U(1)_Y$ gauge symmetry. Both left- and right-handed quarks transform under this $U(1)_Y$ with a hypercharge which is related to the electric charge by the relation eq. (5.54). The left-handed leptons and the e_R also carry hypercharge, but the ν_R has no SM gauge interactions.
- In the symmetry limit (before spontaneous symmetry breaking) the fermions with $SU(2)$ gauge interactions are massless.[13] The spontaneous symmetry breaking mechanism which gives a vev to the scalar field also generates the fermion masses.

[13]This does not apply to ν_R, which *can* have an explicit mass term

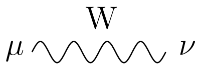
- The scalar multiplet that is responsible for the spontaneous symmetry breaking also carries weak hypercharge. As a result, one neutral gauge boson (the Z) acquires a mass, whereas its orthogonal superposition is the massless photon. The magnitude of the electron charge, e, is then given by $e = g \sin\theta_W$.
- The weak interactions proceed via the exchange of massive charged or neutral gauge bosons. The old four-fermi weak Hamiltonian is an effective Hamiltonian which is valid for low energy processes in which all momenta are small compared with the W mass. The Fermi coupling is obtained in terms of e, M_W and $\sin\theta_W$ by eq. (6.16).

For completeness, a full set of Feynman rules for the case of a single family of leptons is given as an appendix to this lecture.

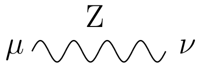
Feynman Rules in the Unitary Gauge (for one Lepton Generation)

Propagators:

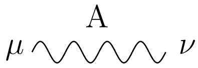
(All propagators carry momentum p.)



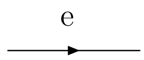
$$-i\,(g_{\mu\nu} - p_\mu\, p_\nu/M_W^2)/(p^2 - M_W^2)$$



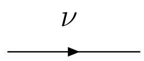
$$-i\,(g_{\mu\nu} - p_\mu\, p_\nu/M_Z^2)/(p^2 - M_Z^2)$$



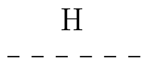
$$-i\,g_{\mu\nu}/p^2$$



$$i\,(\gamma \cdot p + m_e)/(p^2 - m_e^2)$$



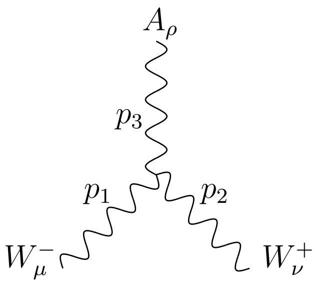
$$i\,\gamma \cdot p/p^2$$



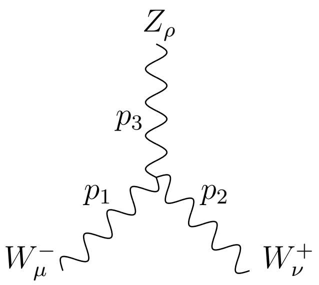
$$i/(p^2 - m_H^2)$$

Three-point gauge-boson couplings:

(All momenta are defined as incoming.)

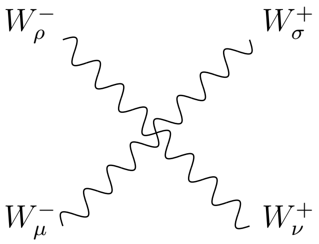


$$i\, g\, \sin\theta_W\, \left((p_1 - p_2)_\rho\, g_{\mu\nu} + (p_2 - p_3)_\mu\, g_{\nu\rho} + (p_3 - p_1)_\nu\, g_{\rho\mu}\right)$$

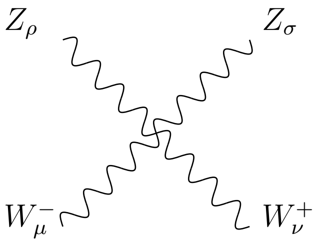


$$i\, g\, \cos\theta_W\, \left((p_1 - p_2)_\rho\, g_{\mu\nu} + (p_2 - p_3)_\mu\, g_{\nu\rho} + (p_3 - p_1)_\nu\, g_{\rho\mu}\right)$$

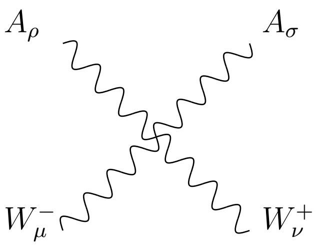
Four-point gauge-boson couplings:



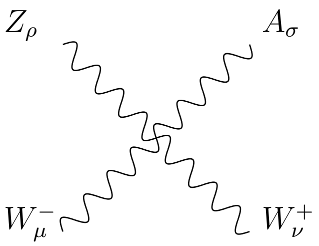
$$i\,g^2\,\left(2g_{\mu\rho}\,g_{\nu\sigma} - g_{\mu\nu}\,g_{\rho\sigma} - g_{\mu\sigma}\,g_{\nu\rho}\right)$$



$$i\,g^2\cos^2\theta_W\,\left(2g_{\mu\nu}\,g_{\rho\sigma} - g_{\mu\rho}\,g_{\nu\sigma} - g_{\mu\sigma}\,g_{\nu\rho}\right)$$

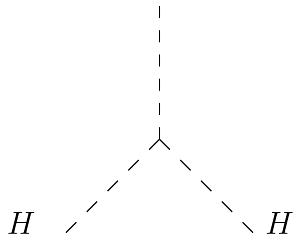


$$i\,g^2\sin^2\theta_W\,\left(2g_{\mu\nu}\,g_{\rho\sigma} - g_{\mu\rho}\,g_{\nu\sigma} - g_{\mu\sigma}\,g_{\nu\rho}\right)$$

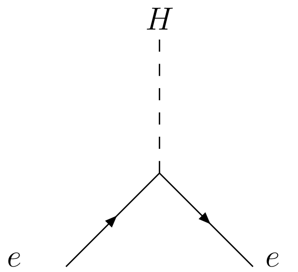


$$i\,g^2\cos\theta_W\,\sin\theta_W\,\left(2g_{\mu\nu}\,g_{\rho\sigma} - g_{\mu\rho}\,g_{\nu\sigma} - g_{\mu\sigma}\,g_{\nu\rho}\right)$$

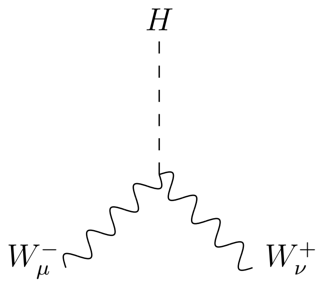
Three-point couplings with Higgs scalars:



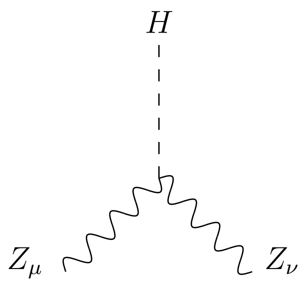
$$-\tfrac{3}{2}\, i\, g\, m_H^2/M_W$$



$$-\tfrac{1}{2}\, i\, g\, m_e/M_W$$

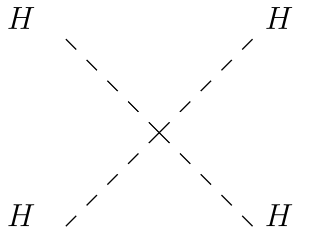


$$i\, g\, M_W\, g_{\mu\nu}$$

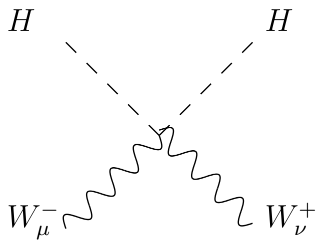


$$i\, (g/\cos^2\theta_W)\, M_W\, g_{\mu\nu}$$

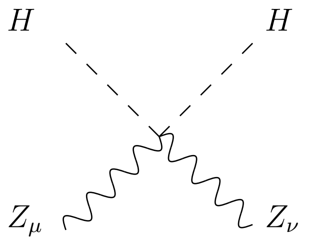
Four-point couplings with Higgs scalars:



$-\frac{3}{4}\, i\, g^2\, (m_H^2/M_W^2)$

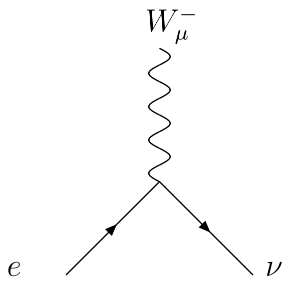


$\frac{1}{2}\, i\, g^2\, g_{\mu\nu}$

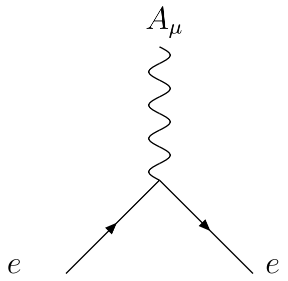


$\frac{1}{2}\, i\, \left(g^2/\cos^2\theta_W\right)\, g_{\mu\nu}$

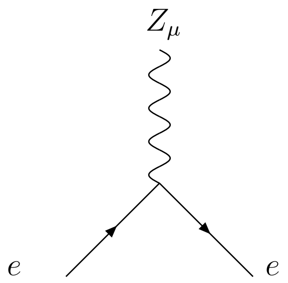
Fermion interactions with gauge bosons:



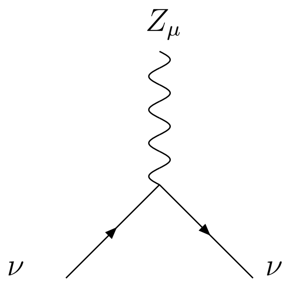
$$-\, i\, \left(g/2\sqrt{2}\right)\, \gamma_\mu\, (1-\gamma^5)$$



$$i\, g\, \sin\theta_W\, \gamma_\mu$$



$$\tfrac{1}{4}\, i\, (g/\cos\theta_W)\, \gamma_\mu\, \left(1 - 4\, \sin^2\theta_W\, - \gamma^5\right)$$



$$-\, \tfrac{1}{4}\, i\, (g/\cos\theta_W)\, \gamma_\mu\, (1-\gamma^5)$$

6 Additional Generations

In the previous section, the Lagrangian of the Standard Model with one family was given. Here we include additional "families" (or "generations") and briefly outline the phenomenological consequences in the quark sector. Family-changing processes among the leptons will be discussed in the neutrino chapter.

6.1 A Second Quark Generation

The second generation of quarks consists of a c ("charm") quark, which has electric charge $+\frac{2}{3}$ and an s ("strange") quark, with electric charge $-\frac{1}{3}$. We can just add a copy of the left-handed isodoublet and copies of the right-handed singlets in order to include this generation.

The only difference would be in the Yukawa interaction terms where the coupling constants are chosen to reproduce the correct masses for the new quarks. But in this case there is a further complication. It is possible to write down Yukawa terms which mix quarks of different generations, e.g. the Yukawa couplings of the previous section become matrices in flavour space,

$$-[Y_d]_{ij}\overline{q_{L}}_i\,\Phi\,d_{Rj} - [Y_u]_{ij}\overline{q_{L}}_i\,\Phi^c\,u_{Rj} \quad + \text{h.c.} \tag{6.1}$$

where i, j are generation indices. The off-diagonal element $[Y_d]_{12}$ seems to gives rise to a mass mixing between d and s quarks.

The Yukawa matrices are $n_f \times n_f$ matrices, where n_f is the number of flavours, and can be diagonalised by independant unitary transformations on the left and right (because $YY^\dagger$ and $Y^\dagger Y$ are hermitian). The physical particles are those that diagonalize the mass matrix. So it is convenient to rotate to the eigenbasis of the mass matrix, where there is *no* Yukawa mixing between quarks of different generations.

Notice that when we add a second generation, it has the same gauge interactions as the first. So if we make a unitary transformation in *generation* space, the fermion kinetic terms remain unchanged. Taking advantage of this freedom, we can rotate u_R, d_R and q_L respectively to the mass eigenstate bases of the u_R, d_R and u_L.

This means, however, that the quark doublets which couple to the gauge bosons *are*, in general, superpositions of physical quarks, because we have written the d_{Li} in the u_{Li} mass eigenstate basis:

$$\begin{pmatrix} u \\ \tilde{d} \end{pmatrix}_L , \tag{6.2}$$

and

$$\left(\begin{array}{c} c \\ \tilde{s} \end{array} \right)_L , \tag{6.3}$$

where $\tilde{d}$ and $\tilde{s}$ are related to the physical d and s quarks by

$$\left(\begin{array}{c} \tilde{d} \\ \tilde{s} \end{array} \right) = \mathbf{V_C} \left(\begin{array}{c} d \\ s \end{array} \right), \tag{6.4}$$

where $\mathbf{V_C}$ is a unitary 2×2 matrix.

Terms which are diagonal in the quarks are unaffected by this unitary transformation of the quarks. Thus the coupling to photons or Z bosons is the same whether written in terms of $\tilde{d}$, $\tilde{s}$ or simply s, d. We will return to this later.

On the other hand the coupling to the charged gauge bosons is

$$-\frac{g}{2\sqrt{2}} \overline{u} \gamma^\mu (1-\gamma^5)\, \tilde{d}\, W_\mu^- \; - \frac{g}{2\sqrt{2}} \overline{c} \gamma^\mu (1-\gamma^5)\, \tilde{s}\, W_\mu^- \quad + \text{h.c.} \tag{6.5}$$

which we may write as

$$-\frac{g}{2\sqrt{2}} \left(\begin{array}{c} \overline{u} \\ \overline{c} \end{array} \right)^T \gamma^\mu (1-\gamma^5)\, \mathbf{V_C} \left(\begin{array}{c} d \\ s \end{array} \right) W_\mu^- \quad + \text{h.c.}. \tag{6.6}$$

The most general 2×2 unitary matrix may be written as

$$\left(\begin{array}{cc} e^{-i\gamma} & \\ & 1 \end{array} \right) \left(\begin{array}{cc} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{array} \right) \left(\begin{array}{cc} e^{i\alpha} & \\ & e^{i\beta} \end{array} \right), \tag{6.7}$$

where we have set one of the phases to 1 since we can always absorb an overall phase by adjusting the remaining phases, α, β and γ.

The phases, α, β, γ can be absorbed by performing a global phase transformation on the d, s and u quarks respectively. This again has no effect on the neutral terms. Thus the most general observable unitary matrix is given by

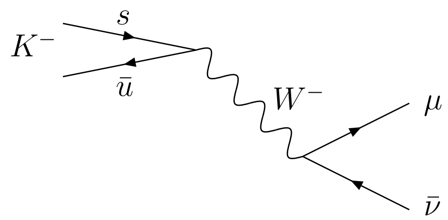
$$\mathbf{V_C} = \left(\begin{array}{cc} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{array} \right), \tag{6.8}$$

where θ_C is the Cabibbo angle.

In terms of the physical quarks, the charged gauge boson interaction terms are

$$\begin{aligned} -\frac{g}{2\sqrt{2}} \Big(& \cos\theta_C\, \overline{u} \gamma^\mu (1-\gamma^5)\, d \;+\; \sin\theta_C\, \overline{u} \gamma^\mu (1-\gamma^5)\, s \\ & + \cos\theta_C\, \overline{c} \gamma^\mu (1-\gamma^5)\, s \;-\; \sin\theta_C\, \overline{c} \gamma^\mu (1-\gamma^5)\, d \Big) W_\mu^- \quad + \text{h.c.}. \end{aligned} \tag{6.9}$$

This means that the u quark can undergo weak interactions in which it is converted into an s quark, with an amplitude that is proportional to $\sin\theta_C$. It is this that gives rise to strangeness violating weak interaction processes, such as the leptonic decay of K^- into a muon and antineutrino. The Feynman diagram for this process is



6.2 Flavour Changing Neutral Currents

Although there are charged weak interactions that violate strangeness conservation, there are no known neutral weak interactions that violate strangeness. For example, the K^0 does not decay into a muon pair or two neutrinos (branching ratio $< 10^{-5}$). This means that the Z boson only interacts with quarks of the same flavour. We can see this by noting that the Z boson interaction terms are unaffected by a unitary transformation. This absence of flavour changing neutral currents (FCNC) in experimental data is rather important. As we will see, in the Standard Model there are no FCNC at tree level, and the absence of FCNC is an important constraint for many extensions of the Standard Model.

The Z boson interactions with d and s quarks are proportional to

$$\overline{\tilde{d}}\,\tilde{d} + \overline{\tilde{s}}\,\tilde{s} \tag{6.10}$$

(we have suppressed the γ-matrices which act between the fermion fields). Writing this out in terms of the physical quarks we get

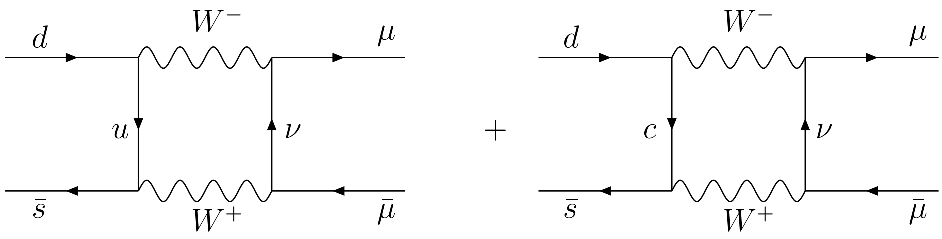
$$\begin{aligned} &\cos^2\theta_C\,\overline{d}\,d \;+\; \sin\theta_C\,\cos\theta_C\,\overline{s}\,d \;+\; \cos\theta_C\,\sin\theta_C\,\overline{d}\,s \;+\; \sin^2\theta_C\,\overline{s}\,s \\ &+\; \cos^2\theta_C\,\overline{s}\,s \;-\; \sin\theta_C\,\cos\theta_C\,\overline{d}\,s \;-\; \cos\theta_C\,\sin\theta_C\,\overline{s}\,d \;+\; \sin^2\theta_C\,\overline{d}\,d\,. \end{aligned} \tag{6.11}$$

We see that the cross terms cancel out and we are left with simply

$$\overline{d}\,d \;+\; \overline{s}\,s\,. \tag{6.12}$$

This cancellation is the reason for the absence of FCNC and is simply a consequence of the unitarity of the mixing matrix eq. (6.7). This effect is also known as the "GIM" (Glashow-Iliopoulous-Maiani) mechanism. It was used to predict the existence of the c quark.

There can be a small contribution to strangeness changing neutral processes from higher order corrections in which we do not exchange a Z boson, but two charged W bosons. The Feynman diagrams for such a contribution to the leptonic decay of a K^0 (which consists of a d quark and an s antiquark) are:



These diagrams differ in the flavour of the internal quark which is exchanged, being a u quark in the first diagram and a c quark in the second. Both of these diagrams are allowed because of the Cabibbo mixing. The first of these diagrams gives a contribution proportional to

$$+\sin\theta_C\ \cos\theta_C\ ,$$

which arises from the product of the two couplings involving the emission of the W bosons. The second diagram gives a term proportional to

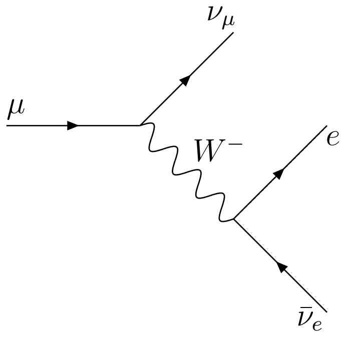
$$-\cos\theta_C\ \sin\theta_C\ .$$

If the c and u quarks had identical masses then these two contributions would cancel precisely. However, because the c quark is much more massive than the u quark, there is some residual contribution. This was used to limit the mass of the c quark to < 5 GeV, before it was discovered.

6.3 Adding Another Lepton Generation

We first neglect the ν_R and neutrino masses. In this approximation, there will be no generation mixing in the lepton sector, so we can include other lepton families, the muon and its neutrino, and the tau-lepton with its neutrino, simply as copies of what we have for the electron and its neutrino. For each family we have a weak isodoublet of left-handed leptons and a right-handed isosinglet for the charged lepton.

Thus, the mechanism which determines the decay of the muon (μ) is one in which the muon converts into its neutrino and emits a charged W^-, which then decays into an electron and (electron-) antineutrino. The Feynman diagram is



The amplitude for this process is given by the product of the vertex rules for the emission (or absorption) of a W^- with a propagator for the W boson between them. Up to corrections of order m_μ^2/M_W^2, we may neglect the effect of the term $q^\mu q^\nu/M_W^2$ in the W-boson propagator, so that we have

$$\left(-i\,\frac{g}{2\sqrt{2}}\,\overline{\nu_\mu}\,\gamma^\rho(1-\gamma^5)\mu\right)\left(\frac{-\,i\,g_{\rho\sigma}}{q^2-M_W^2}\right)\left(-\,i\,\frac{g}{2\sqrt{2}}\,\overline{e}\,\gamma^\sigma(1-\gamma^5)\,\nu_e\right), \qquad (6.13)$$

where q is the momentum transferred from the muon to its neutrino. Since this is negligible in comparison with M_W we may neglect it and the expression for the amplitude simplifies to

$$i\,\frac{g^2}{8M_W^2}\,\overline{\nu_\mu}\,\gamma^\rho(1-\gamma^5)\mu\;\overline{e}\,\gamma_\rho(1-\gamma^5)\nu_e\,. \qquad (6.14)$$

Before the development of this model, weak interactions were described by the "four-fermi model" with a weak interaction Hamiltonian given by

$$\mathcal{H}_{ijkl} \;=\; \frac{G_F}{\sqrt{2}}\,\overline{\psi}_i\,\gamma^\mu(1-\gamma^5)\psi_j\;\overline{\psi}_k\,\gamma_\mu(1-\gamma^5)\psi_l\,. \qquad (6.15)$$

We now recognize this as an effective low-energy Hamiltonian which may be used when the energy scales involved in the weak process are negligible compared with the mass of the W boson. The Fermi coupling constant, G_F, is related to the electric charge, e, the W mass and the weak mixing angle by

$$G_F \;=\; \frac{e^2}{4\sqrt{2}\,M_W^2\,\sin^2\theta_W}. \qquad (6.16)$$

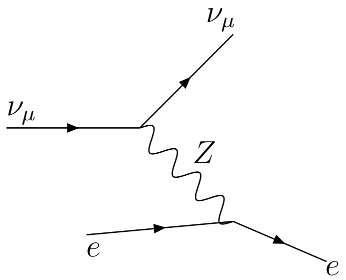
This gives us a value for G_F ,

$$G_F \;= 1.12\times 10^{-5}\ \mathrm{GeV}^{-2}\,, \qquad (6.17)$$

which is very close to the value of 1.17×10^{-5} GeV^{-2} as measured from the lifetime of the muon.

We see that the weak interactions are 'weak', not because the coupling is particularly small (the $SU(2)$ gauge coupling is about twice as large as the electromagnetic coupling), but because the exchanged boson is very massive, so that the Fermi coupling constant of the four-fermi theory is very small. The large mass of the W boson is also responsible for the fact that the weak interactions are short range (of order 10^{-18} m).

In the Standard Model, however, we also have neutral weak currents. Thus, for example, we can have elastic scattering of muon-type neutrinos against electrons via the exchange of the Z boson. The Feynman diagram for such a process is:



Exercise 6.1
Let us write the four-fermi interaction for this process as

$$\mathcal{H} \;=\; \frac{G_F}{\sqrt{2}}\,\overline{\nu_e}\,\gamma^\rho(1-\gamma^5)\nu_e\,\overline{\mu}\,\gamma_\rho(v \;-\; a\gamma^5)\mu\,,$$

where v and a give us the vector and axial-vector coupling of the muon to the Z boson (the muon couples in an identical way to the electron). Determine v and a in terms of θ_W.

6.4 Adding a Third Generation (of Quarks)

Adding a third generation is achieved in a similar way. In this case the three weak isodoublets of left-handed fermions are

$$\begin{pmatrix} u \\ \tilde{d} \end{pmatrix}, \quad \begin{pmatrix} c \\ \tilde{s} \end{pmatrix}, \quad \begin{pmatrix} t \\ \tilde{b} \end{pmatrix}, \tag{6.18}$$

where $\tilde{d}$, $\tilde{s}$ and $\tilde{b}$ are related to the physical d, s and b quarks by

$$\begin{pmatrix} \tilde{d} \\ \tilde{s} \\ \tilde{b} \end{pmatrix} = \mathbf{V_{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{6.19}$$

The 3×3 unitary matrix $\mathbf{V_{CKM}}$ is called Cabibbo-Kobayashi-Maskawa (CKM) matrix. Once again it only affects the charged weak processes in which a W boson is exchanged. For this reason the elements are written as

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{6.20}$$

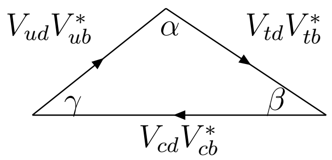
A 3×3 unitary matrix can have nine independent parameters (counting the real and imaginary parts of a complex element as two parameters). In this case there are six possible fermions involved in the charged weak processes and so we can have five relative phase transformations, thereby absorbing five of the nine parameters.

This means that whereas the Cabibbo matrix only has one parameter (the Cabibbo angle, θ_C), the CKM matrix has four independent parameters. If the CKM matrix were real it would only have three independent parameters. This means that in the case of the CKM matrix some of the elements may be complex. The four independent parameters can be thought of as three mixing angles between the three pairs of generations and a complex phase.

The requirement of unitarity puts various constraints on the elements of the CKM matrix. For example we have

$$V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0 .$$

This can be represented as a triangle in the complex plane known as the "unitarity triangle":



The angles of the triangle are related to ratios of elements of the CKM matrix

$$\alpha = -\arg\left\{\frac{V_{td} V_{tb}^*}{V_{ud} V_{ub}^*}\right\}, \tag{6.21}$$

$$\beta = -\arg\left\{\frac{V_{td} V_{tb}^*}{V_{cd} V_{cb}^*}\right\}, \tag{6.22}$$

$$\gamma = -\arg\left\{\frac{V_{ud} V_{ub}^*}{V_{cd} V_{cb}^*}\right\}. \tag{6.23}$$

A popular representation of the CKM matrix is the Wolfenstein parameterisation which uses the parameters A, which is assumed to be of order unity, a complex number $(\rho + i\,\eta)$ and a small number λ, which is approximately equal to $\sin\theta_C$. In terms of these parameters the

CKM matrix is written as

$$\mathbf{V_{CKM}} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\,\lambda^3\,(\rho - i\,\eta) \\ -\lambda & 1-\lambda^2/2 & A\,\lambda^2 \\ A\lambda^3\,(1-\rho-i\,\eta) & -A\,\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)\,. \tag{6.24}$$

We see that whereas the W bosons can mediate a transition between a u quark and a b quark (V_{ub}) or between a t quark and a d quark (V_{td}), the amplitude for such transitions are suppressed by the cube of the small quantity which determines the amplitude for transitions between the first and second generations, λ. The $\mathcal{O}(\lambda^4)$ corrections are needed to ensure the unitarity of the CKM matrix and these corrections have several matrix elements which are complex.

6.5 CP Violation

The possibility that some of the elements of the CKM matrix may be complex provides a mechanism for the violation of CP conservation. Violation of CP conservation has been observed in the $K^0 - \overline{K}^0$ system, and is currently being investigated for B mesons.

Higher-order corrections to the masses of B^0 and $\overline{B^0}$ give rise to mixing between the two states. Thus the mass matrix can be written as

$$\begin{pmatrix} M_{B^0} & \Delta M \\ (\Delta M)^* & M_{B^0} \end{pmatrix}. \tag{6.25}$$

The mass eigenstates are therefore

$$|B_L\rangle = p\,|B^0\rangle + q\,|\overline{B}^0\rangle\,, \tag{6.26}$$

whose mass is $M - \frac{1}{2}\Delta m$, and

$$|B_H\rangle = p\,|B^0\rangle - q\,|\overline{B}^0\rangle\,, \tag{6.27}$$

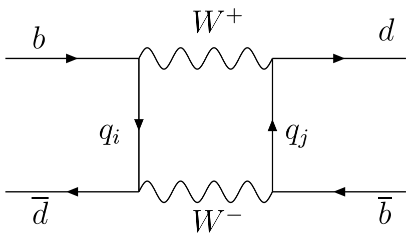
whose mass is $M + \frac{1}{2}\Delta m$, where we have introduced the mass difference between the two mass eigenstates, $\Delta m \equiv 2\sqrt{\Delta M (\Delta M)^*}$.

If ΔM were real then we would have $p = q = 1/\sqrt{2}$ and these mass eigenstates would be CP eigenstates, using the fact that

$$CP|B^0\rangle = -|\overline{B}^0\rangle.$$

However, the non-zero phases in the CKM matrix give rise to a complex phase for ΔM, so that the ratio of p and q is a complex phase, indicating that B_L and B_H are *not* CP eigenstates.

A typical weak interaction contribution to the mass-mixing term, ΔM, is given by the Feynman diagram



Note that on the left we have a B^0, consisting of a b quark and a d antiquark, whereas on the right we have a $\overline{B}^0$ consisting of a d quark and a b antiquark. The internal quarks marked q_i and q_j can each be u, c or t quarks, and each of the vertices carries some element of the CKM matrix. The total contribution, therefore, may be written as

$$\sum_{i=u,c,t} \sum_{j=u,c,t} V_{ib}\, V_{id}^*\, V_{jb}^*\, V_{jd}\, a_{ij} \, .$$

Once again, if all the masses of the quarks were equal then the amplitudes a_{ij} would all be equal, and the sum would vanish by the unitarity constraints imposed on the elements V_{ik}. Since the quarks do not all have the same mass, there is some residual contribution. Indeed, the above diagram is dominated by the term in which a t quark is exchanged on both sides, since this quark is much more massive than the rest.

Restricting ourselves to the t quark exchange contribution, we can read off the phase of this contribution, without calculating the diagram itself. It is given by the phase of the products of the CKM matrix elements entering in the diagram, namely

$$(V_{td}^*\, V_{tb})^2.$$

The phase of this quantity is the square of the ratio of p and q, so we have

$$\frac{p}{q} \;=\; \frac{V_{td}^*\, V_{tb}}{V_{td}\, V_{tb}^*} \, .$$

Now suppose that at time $t = 0$ we prepare a state which is purely B^0. Accounting for the fact that the B^0 meson has a decay rate Γ, we can use eqs. (6.26, 6.27) to write the state at time t as

$$|B(t)\rangle \;=\; e^{-iMt}\, e^{-\Gamma t/2} \left(\cos\left(\frac{\Delta m}{2} t\right) |B^0\rangle \;+\; i\frac{q}{p} \sin\left(\frac{\Delta m}{2} t\right) |\overline{B}^0\rangle \right) . \qquad (6.28)$$

Now suppose that the amplitude for a state $|B^0\rangle$ to decay into some CP eigenstate $|f\rangle$ is A_f, whereas the amplitude for a state $|\overline{B}^0\rangle$ to decay into the state $|f\rangle$ is $\overline{A}_f$. Once again, if

CP were conserved, we would have

$$A_f \;=\; \pm \overline{A}_f \,,$$

but the CP violating phases give rise to a more general complex phase for the ratio of these two amplitudes.

This means that the amplitude to find the state $|f\rangle$ after time t is given by

$$\langle f|\mathcal{H}_{wk}|B(t)\rangle \;=\; e^{-iMt}\,e^{-\Gamma t/2}\left(\cos\left(\frac{\Delta m}{2}t\right)A_f \;+\; i\frac{q}{p}\sin\left(\frac{\Delta m}{2}t\right)\overline{A}_f\right). \tag{6.29}$$

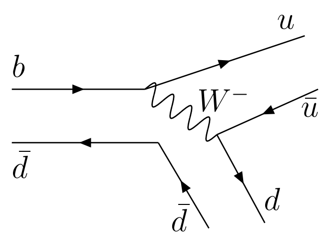
Similarly, if we had prepared a $\overline{B}^0$ at $t=0$ the amplitude to find the state $|f\rangle$ would be

$$\langle f|\mathcal{H}_{wk}|\overline{B}(t)\rangle \;=\; e^{-iMt}\,e^{-\Gamma t/2}\left(\cos\left(\frac{\Delta m}{2}t\right)\overline{A}_f \;-\; i\frac{p}{q}\sin\left(\frac{\Delta m}{2}t\right)A_f\right). \tag{6.30}$$

Taking the moduli squared for the decay rates we derive the result

$$\frac{\Gamma(B(t)\to f)-\Gamma(\overline{B}(t)\to f)}{\Gamma(B(t)\to f)+\Gamma(\overline{B}(t)\to f)} \;=\; -\;\sin\left(\Delta m\,t\right)\Im m\left(\frac{q}{p}\frac{\overline{A}_f}{A_f}\right). \tag{6.31}$$

For example, if the state $|f\rangle$ is the CP even two-pion state $|\pi^0\,\pi^0\rangle$, the Feynman diagram at the quark level for $A_{2\pi}$ is



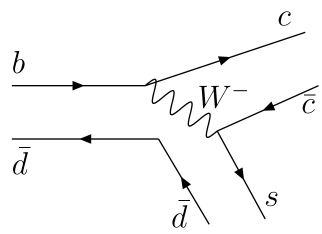
To fully calculate the decay amplitudes we would need to know the wave functions for the mesons in terms of the constituent quark-antiquark pairs, but for the ratio $\overline{A}_{2\pi}/A_{2\pi}$ we just need the ratios of the CKM matrix elements occurring in this diagram, namely

$$\frac{\overline{A}_{2\pi}}{A_{2\pi}} \;=\; \frac{V_{ub}\,V_{ud}^*}{V_{ub}^*\,V_{ud}}\,,$$

so that (using eq. (6.21))

$$\Im m\left(\frac{q}{p}\frac{\overline{A}_{2\pi}}{A_{2\pi}}\right) \;=\; \frac{V_{td}\,V_{tb}^*}{V_{td}^*\,V_{tb}}\frac{V_{ub}\,V_{ud}^*}{V_{ub}^*\,V_{ud}} \;=\; -\sin\left(2\alpha\right). \tag{6.32}$$

As a further example we consider the so-called "golden channel" where $|f\rangle$ is the state $|J/\psi\,K_S\rangle$. In this case the quark level Feynman diagram is



Here there is a further complication since the outgoing state $(s\,\bar{d})$ is actually a $\overline{K}^0$ (and likewise for the $\overline{B}^0$ decay it would be a K^0). As in the B^0 system, the mass eigenstates are given by

$$\begin{aligned} |K_S\rangle &= p_K|K^0\rangle + q_K|\overline{K}^0\rangle\,, \\ |K_L\rangle &= p_K|K^0\rangle - q_K|\overline{K}^0\rangle\,. \end{aligned} \tag{6.33}$$

Once again, if CP were conserved we would have $p_K = q_K = 1/\sqrt{2}$, and these mass eigenstates would be eigenstates of CP. The phases in the CKM matrix introduce a phase in the ratio of p_K and q_K, calculated from diagrams similar to the ones for the B^0 system (but with the b quark replaced by an s quark). In this case it is the diagram with an internal c quark exchange that dominates (although the mass of the c is much smaller than the t quark mass, the CKM matrix elements are much larger for c quark exchange than for t quark exchange and this effect dominates), so we have a factor

$$\frac{q_K}{p_K} = \frac{V_{cd}^*\,V_{cs}}{V_{cd}\,V_{cs}^*}$$

which enters in the ratio of the decay amplitudes, giving

$$\frac{\overline{A}_{J/\psi\,K_S}}{A_{J/\psi\,K_S}} = -\frac{V_{cb}\,V_{cs}^*}{V_{cb}^*\,V_{cs}}\frac{V_{cd}^*\,V_{cs}}{V_{cd}\,V_{cs}^*} = -\frac{V_{cb}\,V_{cd}^*}{V_{cb}^*\,V_{cd}}$$

(a minus occurs because the $J/\psi\,K_S$ state is CP odd), so that (using eq. (6.22))

$$\Im m\left(\frac{q}{p}\frac{\overline{A}_{J/\psi\,K_S}}{A_{J/\psi\,K_S}}\right) = -\frac{V_{td}\,V_{tb}^*}{V_{td}^*\,V_{tb}}\frac{V_{cb}\,V_{cd}^*}{V_{cb}^*\,V_{cd}} = \sin(2\beta)\,. \tag{6.34}$$

6.6 Summary

- Additional generations may be added, with gauge interactions copied from the first, but in this case one can have mass-mixing between quarks of different generations. In terms of the mass eigenstates, the charged W bosons mediate transitions between a $T^3 = +\frac{1}{2}$ quark (u, c or t) and a superposition of $T^3 = -\frac{1}{2}$ quarks (d, s and b). In two generations, this mechanism allows weak interactions that violate strangeness conservation, and the mixing matrix has only one independent parameter, the Cabibbo angle.

- The unitarity of the mixing matrix guarantees that there are no strangeness changing neutral processes. Weak interactions involving the exchange of a Z boson do not change flavour. There is a small violation of this in higher orders owing to the mass splitting between the quarks.

- Including a third generation, the mixing matrix for the $T^3 = -1/2$ quarks (d, s and b) is the CKM matrix. This matrix has four independent parameters, so that some of the matrix elements may be complex.

- The possibility that some of the elements of the CKM matrix may be complex leads to a weak interaction contribution to the mass mixing of B^0 and $\overline{B}^0$ which can be complex. This gives rise to CP violation, since the eigenstates of the B^0 mass matrix are then no longer eigenstates of CP. The CKM matrix also introduces phases in the ratios of the decay amplitudes for B^0 and $\overline{B}^0$ to a given CP eigenstate. Products of the phase of the mass mixing and the ratio of the decay amplitudes can be observed directly in tagged B meson experiments, and the angles α and β of the unitarity triangle can be directly measured.

7 Neutrinos

In its original formulation, the Standard Model had massless neutrinos — neutrino masses were not measured at the time. We now know that neutrinos have a (very small) mass, which can be accommodated in the SM in a straightforward way. We will discuss this in the second part of this chapter. There are two possible types of neutrino mass terms, "Dirac" and "Majorana", because the neutrino has zero electric charge. This makes neutrino mass terms a bit different from those of the other fermions and may explain why neutrinos are much lighter than SM fermions.

In the first part of this chapter we focus on the currently observed consequence of small neutrino masses, neutrino oscillations.

7.1 Neutrino Oscillations

Recall that in the quark sector, there were flavour changing charged current processes, that is, the W could interact with an up-type quark of one generation, and a down-type quark of another. If the neutrinos have mass, we should get exactly the same effect in the lepton sector, except that the mixing matrix U_{fm} is called the PMNS matrix (for Pontecorvo, Maki, Nakagawa and Sakata), rather than CKM. The index order "flavour-mass" in U_{fm} indicates that U rotates a vector from the neutrino mass basis to the neutrino "flavour" basis, which is the charged lepton mass basis.

The physical consequences of mixing angles are quite different between the lepton sector and the quarks. This is because neutrinos are very light and have only weak interactions. In the quark sector one can differentiate $D \to K\bar{\mu}\nu$ from $D \to \pi\bar{\mu}\nu$, because the π and K have strong and electromagnetic interactions, which allows us to track them in the detector, and they have sufficiently different masses that the tracks are distinguishable. This is not the case in trying to distinguish $\mu \to e\nu_3\bar{\nu}_2$ from $\mu \to e\nu_3\bar{\nu}_1$.

The small masses and weak interactions of neutrinos imply that the wave packets corresponding to different neutrino mass eigenstates remain superposed over long distances. The effects of flavour mixing can therefore be seen via oscillations.

For simplicity we will consider the case of two generations which in the charged lepton sector we will take to be the electron and muon.[14] We label the neutrino mass eigenstates as ν_1

[14] Of course, in the Standard Model we have three families, but the important concepts can be understood in the simpler case.

and ν_2. They are related by an equation very similar to eq. (6.4),

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} . \tag{7.1}$$

Now we would like to compute the amplitude for an oscillation process. Suppose that we have an initial beam of muons which decays to relativisitic neutrinos of energy E and momentum k. The neutrinos travel a distance $L = \tau$ to a detector where they produce an e or a μ by charged current (CC) scattering. The amplitude will be

$$\mathcal{A}_{\mu\alpha} \sim \sum_j U_{\mu j} \times e^{-i(E_j\tau - k_j L)} \times U^*_{\alpha j} \,, \tag{7.2}$$

where the three pieces arise from production, propagation and detection. (From your field theory notes, you can check that the Feynman propagator in position space, $G(0, (\tau, L))$, is the exponential, where the momentum integral in the propagator was taken care of in the production process of the neutrinos with 4-momentum (E, k).)

First, suppose that we can neglect the neutrino masses, so $(E_j, k_j) = (E_n, k_n)$ for any j, n. The propagation exponential can then be factored out, and (7.2) is the unitarity condition for U,

$$U_{\mu j} U^*_{\alpha j} = \delta_{\mu\alpha} \,. \tag{7.3}$$

Recall that for quarks, with three generations, this relation gives the unitarity triangle.

Now we allow the neutrinos to have small masses, $m \ll E, k$, so that $L \simeq \tau$ remains. Then the exponent can be written as

$$-i(E_j\tau - k_j L) \;\simeq\; -i(E_j - k_j)L \;=\; -i\frac{E_j^2 - k_j^2}{E + k}L \;\simeq\; -i\frac{m_j^2}{2E}L \,, \tag{7.4}$$

such that

$$\mathcal{P}_{\mu\alpha} \;=\; |\mathcal{A}_{\mu\alpha}|^2 \;=\; |\sum_j U_{\mu j} e^{-i\Delta m_j^2 L/(2E)} U^*_{\alpha j}|^2 \,. \tag{7.5}$$

Using the explicit form of U given in eq. (7.1) one obtains the muon survival probability

$$\mathcal{P}_{\mu\mu} \;=\; 1 - \sin^2 2\theta \sin^2 \frac{(m_2^2 - m_1^2)L}{4E} \,. \tag{7.6}$$

In reality, there are three generations of leptons in the SM, so the MNS matrix U is 3×3, and there are three mass eigenstates in the sum of eq. (7.5). As in the case of CKM, MNS can be written in terms of three angles and one phase:

$$\hat{U} \;=\; \begin{bmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{bmatrix} \tag{7.7}$$

$$\simeq \left[\begin{array}{ccc} c_{12} & s_{12} & s_{13}e^{-i\delta} \\ -s_{12}/\sqrt{2} & c_{12}/\sqrt{2} & 1/\sqrt{2} \\ s_{12}/\sqrt{2} & -c_{12}/\sqrt{2} & 1/\sqrt{2} \end{array} \right], \tag{7.8}$$

where the "solar" angle $\theta_{12} \simeq \pi/6$, and we have used the approximate measured value of the atmospheric angle $\theta_{23} \simeq \pi/4$. $s_{13} = \sin\theta_{13} \leq 0.2$ is known from experimental bounds and θ_{13} is significantly smaller than the other two angles. Note that, unlike the quark sector, some mixing angles are large. Combined with the small neutrino masses, this is puzzling and provoking to theorists, who expend much effort into building models of this.

It is often said that MNS has three phases, so let us recall the phase choices that allow us to write eq. (7.7), so as to understand where the other two phases could be:

- A 3×3 complex matrix has 18 real parameters.
- The unitarity condition $UU^\dagger = 1$ reduces this to 9, which can be taken as 3 angles and 6 phases.
- Five of those phases are relative phases between the fields e, μ, τ, ν_1, ν_2 and ν_3,
- ... so if we are free to choose the phases of all the left-handed fermions, we are left with one phase in the mixing matrix. This was the case with the quarks, where any potential phase in the quark masses could be absorbed by the right-handed fermion fields.
- If the right-handed fields do not appear in our physical process (which means the masses appear as mm^*), then we are free to make the above phase choice, and our process is independant of any possible phase of the masses. This is the case for neutrino oscillations.
- We will see in a later section that the ν_L can have so-called "Majorana" masses, between themselves and their antiparticle. This means that it is the left-handed neutrino field which must absorb the phase of the Majorana mass. So in physical processes where the Majorana mass appears linearly (and not as mm^*; this is the case e.g. in neutrinoless double-beta decay), one can choose the phase such that the mass is real — in which case one can remove one less phase from MNS, or one can keep MNS with one phase, and allow complex masses.
- It is always possible to remove the phase from one majorana mass by using the global overall phase of all the leptons. (This overall phase corresponds to the global symmetry of lepton number conservation in a theory without majorana masses and is the sixth

phase of e, μ, τ, ν_1, ν_2 and ν_3, which we could not use to remove phases from the lepton number conserving MNS matrix.) So, in three generations, there are possibly two complex majorana neutrino masses, so two "Majorana" phases in addition to the "Dirac" phase δ of MNS.

Although there are three generations, it is well known that for the oscillation probabilities we observe, with the mixing angles that are measured, two neutrino probabilities are a very good approximation. Why is this?

Let us return to the oscillation amplitude $\mathcal{A}_{\alpha\beta}(L)$, and imagine it as the sum of three vectors in the complex plane. If $\alpha = \beta$, the unitarity condition at $L = 0$ says they should sum to a vector of length one. If $\alpha \neq \beta$, then they should sum to zero and this is the unitarity triangle. At non-zero L, two of the vectors rotate in the complex plane, with frequencies $(m_j^2 - m_1^2)/2E$ — so neutrino oscillations correspond, in some sense, to time-dependent non-unitarity.

Consider the oscillation probabilities $\mathcal{P}_{\mu\alpha}$, measured for atmospheric neutrinos, on length scales corresponding to $m_3^2 - m_1^2$. The solar mass difference can be neglected, because $m_2^2 - m_1^2 \ll m_3^2 - m_1^2$, so there is only one relevant mass difference, and the survival probability behaves as for two generations. This is easy to visualise in the complex plane, where only the vector $U_{\mu 3}U_{\alpha 3}^*$ rotates with L. The stationary sum $U_{\mu 2}U_{\alpha 2}^* + U_{\mu 1}U_{\alpha 1}^*$ can be treated as a single vector, so this looks like a two generation system. So "atmospheric" oscillations can be approximated as two-neutrino oscillations because the atmospheric mass difference is very large compared to the solar one.

In the case of the solar mass difference, measured for instance at KamLAND, the two neutrino approximation is good because θ_{13} is small. The observed survival probability is $\mathcal{P}_{ee}$ and since $U_{e3} \ll U_{ej}$, $j = 1, 2$, the last term can be dropped in

$$\mathcal{A}_{ee} = \sum_j U_{ej} e^{-i\Delta m_j^2 L/(2E)} U_{ej}^* \,. \tag{7.9}$$

7.2 Oscillations in Quantum Mechanics (in Vaccum and Matter)

This subsection reviews a more conventional derivation of neutrino oscillations in two generations, and includes neutrino oscillations in matter. Electron neutrinos acquire an effective mass term from their interactions with dense matter — this is the MSW effect — which can have significant effects in the sun and in supernovae, and over long baselines in the earth.

In the mass eigenbasis we have the Schrödinger equation

$$i\frac{d}{dt}\Psi = H \cdot \Psi \tag{7.10}$$

with a diagonal Hamiltonian

$$H = \left(\begin{array}{cc} E_1 & 0 \\ 0 & E_2 \end{array} \right) . \tag{7.11}$$

This Schrödinger equation can easily be solved. Defining our initial states at $t = 0$ as $|1\rangle \equiv |1(t=0)\rangle$, $|2\rangle \equiv |2(t=0)\rangle$ we get the time dependent states

$$\begin{aligned} |1(t)\rangle &= e^{-iE_1 t}|1\rangle \, , \\ |2(t)\rangle &= e^{-iE_1 t}|2\rangle \, . \end{aligned} \tag{7.12}$$

Let us repeat the last few steps in the interaction eigenbasis. Multiplying eq. (7.10) by V from the left we get the corresponding Schrödinger equation as

$$i\frac{d}{dt}\tilde{\Psi} = \tilde{H} \cdot \tilde{\Psi} \tag{7.13}$$

with

$$\tilde{H} \equiv V \cdot H \cdot V^{-1} = \left(\begin{array}{cc} a+b & c \\ c & a-b \end{array} \right) , \tag{7.14}$$

where

$$a = \frac{1}{2}(E_1 + E_2) \, , \tag{7.15}$$

$$b = \frac{1}{2}(E_1 - E_2) \cos(2\theta) \, , \tag{7.16}$$

$$c = -\frac{1}{2}(E_1 - E_2) \sin(2\theta) \, . \tag{7.17}$$

The crucial feature of the new Hamiltonian is that it is no longer diagonal. As a result, if we start at time $t = 0$ with an interaction eigenstate $|\alpha\rangle$, then at a later time we get a superposition of $|\alpha\rangle$ and $|\beta\rangle$ interaction eigenstates. Indeed, using eq. (7.1) for the time dependent states we get

$$|\alpha(t)\rangle = e^{-iE_1 t} \cos\theta \, |1\rangle + e^{-iE_2 t} \sin\theta \, |2\rangle \, , \tag{7.18}$$

$$|\beta(t)\rangle = -e^{-iE_1 t} \sin\theta \, |1\rangle + e^{-iE_2 t} \cos\theta \, |2\rangle \, . \tag{7.19}$$

Let us now use this relation to compute the oscillation probability $\mathcal{P}_{\alpha\to\beta}(t)$. What we mean by this is the following: assume that at $t = 0$ we know that our state is a pure interaction eigenstate $|\alpha\rangle$. To be concrete we can assume this is an electron neutrino ν_e created in the sun. $\mathcal{P}_{\alpha\to\beta}(t)$ then gives us the probability that at a later time t this state has evolved into an interaction eigenstate $|\beta\rangle$. Of course, this probability is simply the absolute value of the amplitude squared

$$\mathcal{P}_{\alpha\to\beta}(t) = |\langle\beta|\alpha(t)\rangle|^2$$

$$
\begin{aligned}
&= \left| -\sin\theta\,\cos\theta \left(e^{-iE_1 t} - e^{-iE_2 t} \right) \right|^2 \\
&= \frac{1}{2}\sin^2(2\theta)\,(1 - \cos(E_2 - E_1)t) \\
&= \sin^2(2\theta)\sin^2\left(\frac{E_2 - E_1}{2}\, t \right) . \qquad (7.20)
\end{aligned}
$$

In the first step we have used eq. (7.18) and the orthogonality of the mass eigenstates $\langle i|j\rangle = \delta_{ij}$. The expression for $\mathcal{P}_{\alpha\to\beta}(t)$ can be brought into a more useful form by noting that

$$
E_i = \sqrt{p^2 + m_i^2} \;=\; p + \frac{m_i^2}{2p} + \ldots \qquad (7.21)
$$

and, therefore,

$$
\frac{1}{2}(E_2 - E_1) \;\simeq\; \frac{m_2^2 - m_1^2}{4E} \;\equiv\; \frac{\Delta m^2}{4E} \qquad (7.22)
$$

where E is the energy of the beam.[15] Furthermore, since the neutrinos travel at the speed of light, we have $L = vt = ct = t$, where L is the distance travelled by the neutrino. Thus, we arrive at the final expression for the oscillation probability,

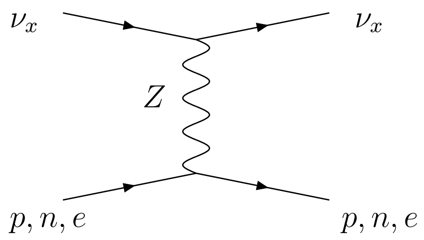
$$
\mathcal{P}_{\alpha\to\beta}(t) = \sin^2(2\theta)\sin^2\left(L\,\frac{\Delta m^2}{4E} \right) . \qquad (7.23)
$$

Eq. (7.23) has the expected properties in that the probability vanishes for $L \to 0$, $\theta \to 0$ and most notably for $\Delta m^2 \to 0$. This last limit tells us that there is no mixing if the two neutrino species have the same mass and, in particular, if they are massless.

So far we have considered oscillations in vacuum, i.e. we have assumed that the neutrinos were travelling through the vacuum. While this is true most of the time, the neutrinos produced in the sun first have to travel through the sun before they can reach us. The matter surrounding the neutrinos can have a crucial effect on the oscillation probability for the neutrinos. This effect is called the matter effect or the Mikheyev-Smirnov-Wolfenstein (MSW) effect.

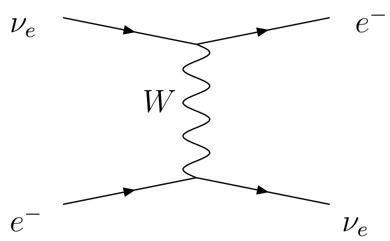
The question at the heart of the problem is: how does the Hamiltonian $\tilde{H}$, eq. (7.14), change through interactions of the neutrinos with surrounding matter? There are basically neutral and charged current interactions. As we have learnt, neutral current interactions are mediated by the exchange of a Z boson. Taking into account that the surrounding matter is basically made of protons, neutrons and electrons, a typical Feynman diagram is:

[15] This argument can be made more rigorously using wave packets.



The important point is that these interactions are independent of the flavour x of the neutrino. Thus they affect the two diagonal entries of the Hamiltonian in the same way. This means they change a, eq. (7.15), i.e. the Hamiltonian is modified by $a \to \bar{a}$. As we will see later, this change is irrelevant.

The charged current interactions are mediated by a $W^{\pm}$. A typical Feynman diagram is:



These interactions take place only for electron neutrinos since there are no μ's (or τ's) in the surrounding matter. In our convention where we identify the $|\alpha\rangle$ state with an electron neutrino, this means that only the top-left entry of the Hamiltonian, eq. (7.14), is modified. Thus, including the matter effects we arrive at the following Hamiltonian,

$$\tilde{H}_{\text{MSW}} = \begin{pmatrix} \bar{a} + b + w & c \\ c & \bar{a} - b \end{pmatrix}, \tag{7.24}$$

where w comes from the charged current interactions. The explicit form of w is not important for us. What we want to know is how the w-term modifies the mixing angle. To find the modified mixing angle θ_{MSW} we have to diagonalize $\tilde{H}_{\text{MSW}}$, i.e. we have to find

$$V_{\text{MSW}} = \begin{pmatrix} \cos\theta_{\text{MSW}} & \sin\theta_{\text{MSW}} \\ -\sin\theta_{\text{MSW}} & \cos\theta_{\text{MSW}} \end{pmatrix} \tag{7.25}$$

such that

$$H_{\text{MSW}} \equiv V_{\text{MSW}}^{-1} \cdot \tilde{H}_{\text{MSW}} \cdot V_{\text{MSW}} \tag{7.26}$$

is diagonal. If we plug the explicit forms for V_{MSW}, eq. (7.25), and $\tilde{H}_{\mathrm{MSW}}$, eq. (7.24), into eq. (7.26) we find the off-diagonal terms of H_{MSW} to be

$$c\cos(2\theta_{\mathrm{MSW}}) + \frac{2b+w}{2}\sin(2\theta_{\mathrm{MSW}})\,. \tag{7.27}$$

This vanishes for

$$\tan(2\theta_{\mathrm{MSW}}) \;=\; -\frac{2c}{2b+w} \;=\; \frac{-\Delta m^2 \sin(2\theta)}{4Ew - \Delta m^2\cos(2\theta)} \tag{7.28}$$

where we have used eqs. (7.16) and (7.17).

We note that θ_{MSW} does not depend on a, thus as mentioned above, the change $a \to \bar{a}$ induced by the neutral current interactions does not matter at all. The important point is that for $4Ew \sim \Delta m^2\cos(2\theta)$ there can be a dramatic effect and the oscillation probability can increase substantially. In fact, this effect is very important in the explanation of experimental results.

7.3 The See-Saw Mechanism

In this section we are concerned with neutrino masses and offer a possible explanation as to why they might be so small compared to other fermion masses. We will restrict ourselves to the case of one family.

As mentioned previously, introducing a right-handed neutrino allows us to write down the same kind of Yukawa coupling as for the u-type quarks, eq. (5.25). This will result in a 'usual' Dirac mass term for the neutrinos of the form

$$m_D\,\overline{\nu}\nu = m_D(\overline{\nu}_L\nu_R + \overline{\nu}_R\nu_L) \tag{7.29}$$

(compare to eq. (5.27)). There is no doubt that such a term can be introduced in the Lagrangian, but it leads immediately to the question of why the ν mass is so much smaller than the other fermion masses. In fact, we would expect that the Yukawa couplings of all fermions are roughly of the same order. This would lead to neutrino masses roughly of the same size as the masses of the other leptons, obviously in sharp contrast to observations.

However, the very special properties of the right-handed neutrinos allow us to write down yet another term in the Lagrangian. Recall that we want to write down the most general gauge invariant Lagrangian, given the gauge group and the matter content. In fact, since ν_R is a singlet under all gauge transformations, we can (or even have to) add a term like

$$M\nu_R\nu_R \;+\; \text{h.c.}\,. \tag{7.30}$$

Note that for this term to be gauge invariant it is mandatory that $Y(\nu_R) = 0$ and that ν_R neither couples to $SU(2)$ nor $SU(3)$ gauge bosons.

This is a Majorana mass term, but its fermion index contraction is perhaps unclear, so let us consider this with some care:

- The Dirac mass for a four-component spinor is of the form

$$m_D \bar{\psi}\psi \;=\; m_D \psi^\dagger \gamma_0 (P_L^2 + P_R^2)\psi \;=\; m_D(\overline{\psi_R}\psi_L + \overline{\psi_L}\psi_R)\,. \tag{7.31}$$

  So to get a Lorentz scalar, a left-handed two-component fermion must be contracted with a rigth-handed two-component fermion.

- Recall that the antiparticle of a chiral fermion has opposite chirality from the particle:
  1) The negative energy solutions of momentum $\vec{p}$ became the positive energy solutions of $-\vec{p}$.
  2) For a massless (= chiral) particle, helicity = chirality, and helicity is $\vec{s}\cdot\vec{p}$, so the antiparticle has opposite chirality from the particle.

In analogy with the Dirac mass term, one could try to write a mass term between the chiral ψ_L and its antiparticle as

$$m\,\overline{(\psi_L)^c}\,\psi_L \;\; + \text{h.c.}\,. \tag{7.32}$$

One should take care with such expressions in the literature, because the operations $^-$, C and P_L do not commute, and different authors perform them in different order. Eq. (7.32) is a Lorentz scalar and can also be expressed as $m\psi_L^T i\sigma_2 \psi_L$ and is often written as $m\psi_L\psi_L$, with the index contraction understood. This is the notation of eq. (7.30).

Whereas m_D is expected to be of the same size as charged-lepton masses, the most natural value for M is much larger. Ultimately we expect that at a high energy scale (maybe the GUT scale $M \sim 10^{15}$ GeV) there is a theory that explains all of the fermion masses. Then, the natural value for the fermion masses is of the order M. However, all fermion masses except for the ν_R are 'protected' by chiral symmetry. This explains why $m_D \ll M$. To understand the consequences of $M \gg m_D$ consider the neutrino mass matrix

$$\begin{pmatrix} \overline{(\nu_L)^c} & \overline{\nu_R} \end{pmatrix} \begin{bmatrix} 0 & m_D \\ m_D & M \end{bmatrix} \begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix}. \tag{7.33}$$

In order to get the masses of the physical particles, i.e. the eigenstates of the mass matrix, we have to diagonalize this matrix. The eigenvalues are approximately given by

$$\frac{m_D^2}{M} \quad \text{and} \quad M\,, \tag{7.34}$$

where we used $m_D \ll M$. Thus we can see that the physical neutrinos are a (nearly) left-handed neutrino with mass m_D^2/M and a (nearly) right-handed neutrino with mass M. Taking $m_D \sim m_t$ and $M \sim 10^{15}$ GeV, we get $m \sim 0.03$ eV, which is not too far from the measured atmospheric mass difference. This may serve as an explanation as to why the mass of the left-handed neutrino is so much smaller than the mass of the other leptons.

If this explanation is correct, then there should also be very heavy (nearly sterile) right-handed neutrinos. If they have GUT-scale masses, they may not be interesting for collider experiments, but they can be relevant in cosmology. If the ν_R are produced in the universe after inflation, they could produce a lepton asymmetry in their decay. The Standard Model has non-perturbative B+L violating interactions, which are rapid at temperatures $T > m_W$, which would partially transform this lepton asymmetry into a baryon asymmetry. This scenario, called leptogenesis, appears to work (it may require CP violation beyond the SM) and adds to the attraction of the seesaw model.

7.4 Summary

- When neutrino masses are included in the Lagrangian, mixing angles appear at the charged current vertex, as in the quark case.
- The experimental signature of (small) neutrino masses is oscillations: a neutrino produced from one flavour of charged lepton, can be detected by the appearance of a different charged lepton. Thus, an electron neutrino produced in the sun can arrive as a neutrino of a different flavour on earth.
- If the neutrinos travel through matter rather than the vacuum the oscillation pattern can change dramatically.
- The see-saw mechanism provides us with an explanation of why the neutrino masses are so much smaller than the other lepton masses.

8 Supersymmetry

This is the only section truly beyond the Standard Model. However, supersymmetry (SUSY) plays an important role in particle physics phenomenology, so in this section we will outline the basic ideas of this new symmetry, why so many theorists like it and sketch how to 'supersymmetrise' the Standard Model.

Supersymmetry is a big topic, and this is a short lecture. There are books and review articles for readers of all tastes. In preparing this lecture, I have used, among others, a phenomenological introduction by S. Martin, `hep-ph/9709356` ($\sim$ 100 pages) — which uses the space-time metric $(-,+,+,+)$, and also a review of physics beyond the Standard Model (BSM) by M. Peskin, `hep-ph/9705479`.

8.1 Why Supersymmetry?

We have learned from LEP and other experiments that loop calculations work. This is a shining success for the Standard Model: we calculate, as a function of a few input parameters, quantum corrections to many (precision) observables, and what is measured agrees very well with the calculations. Nevertheless, there are several arguments as to why the Standard Model is probably not valid for energies up to the GUT scale.

First of all, the Standard Model requires a 'light' Higgs boson of mass $\sim$ 100 GeV. However, if one calculates loop corrections to the Higgs boson mass, they are "quadratically divergent", that is proportional to Λ_{NP}^2 where Λ_{NP} is the scale of New (BSM) Physics. There are various conclusions that one can draw: there is new physics close to the electroweak scale that does not contribute visibly in the precision observables of LEP, or the loop contributions cancel against each other, or the Higgs mass in the Lagrangian has just the right value to cancel the quadratic divergences (this is called "fine tuning", and unpopular not only among theorists). We will see that supersymmetry is a combination of the first and the second solution.

Secondly, the running of the gauge couplings indicates that, at a very high energy scale, the strong, electromagnetic and weak interactions may combine into one unified force, with one unique coupling strength. Within the SM, this GUT scenario does not quite work, but it can be achieved within supersymmetric extensions due to the additional particle content which contributes to the running of the couplings.

Thirdly, even though the SM works amazingly well in the sector of electroweak precision observables, one of the most precise tests of all fails by about $3.4\,\sigma$. The measurement of the anomalous magnetic moment of the muon, $g-2$, from BNL, is larger than the SM prediction.

This discrepancy could well be solved within supersymmetry, but less easily (or not at all) in other extensions of the SM.

In addition to the above arguments, SUSY could also supply a much sought after dark matter candidate, e.g. with the neutralino as the lightest stable neutral SUSY particle.

In the following we will give a brief introduction into the formalism and consequences of SUSY.

Supersymmetry is a transformation which turns bosons into fermions, and fermions into bosons. If it is a symmetry of the Lagrangian, then every fermion must have a bosonic partner and vice versa, and the interactions are restricted by the symmetry. When we supersymmetrise (exactly) the Standard Model, we will therefore (more than) double the number of particles — but the number of coupling constants stays (almost) the same.

Exercise 8.1
Consider the interaction Lagrangian

$$\mathcal{L} \;=\; y_f H(\overline{t_L} t_R \;+\; \overline{t_R} t_L) + \frac{y_s^2}{2} H^2 (T_1 T_1^* + T_2 T_2^*)$$

where t_L, t_R are chiral fermions (the top?), H is a real scalar and T_1 and T_2 are complex scalars.

- Draw the Feynman diagrams for the one-loop contributions to the Higgs mass from t, T_1 and T_2.
- Using Feynam rules from the lectures, calculate the leading (= most divergent) part of the diagrams at zero external momentum.
- Find a desirable relation between y_f and y_S, such that the divergences cancel.
- Now include soft scalar masses

$$\delta\mathcal{L} \;=\; m_T^2 (T_1 T_1^* + T_2 T_2^*)\,,$$

take the supersymmetric relation that you have found between y_f and y_s, and estimate the same one-loop diagrams.

8.2 A New Symmetry: Boson ↔ Fermion

Recall that a symmetry, be it local gauge, or global like Poincaré, is defined by operators which generate the transformations under which the Lagrangian transforms to itself, plus a total divergence. These operators are called generators.

We are looking for an operator Q, acting on bosons $|b\rangle$ and fermions $|f\rangle$ such that

$$Q|b\rangle = |f\rangle, \qquad Q|f\rangle = |b\rangle\,. \tag{8.1}$$

Bosons have even spin and mass dimension (where I am counting the mass dimension of a field in four dimensions), fermions have odd spin and mass dimension, so we conclude that the operator Q should have spin 1/2 and mass dimension 1/2. And since it transforms bosons into fermions, and fermions into bosons, our supersymmetric Lagrangian should have exactly the same number of fermionic and bosonic degrees of freedom. So there is a complex scalar for every chiral fermion, a chiral fermion for each massless vector, and fundamental real scalars are not allowed.

Since Q is a fermion, it should have a spinor index. By statistics and dimensional analysis, we can imagine it acting on fields (operators) as

$$\begin{aligned} &[Q^\alpha, \phi] \sim \psi^\alpha\,, \\ &\{Q^\alpha, \psi\} \sim \partial_\mu \phi + m\phi + g\phi^2, \ldots A_\mu\,. \end{aligned} \tag{8.2}$$

It is clear that Q^α changes spin, so mixes into the Poincaré group of translations and rotations. It can be shown that there is one way, and only one way, of extending the commutation relations of the Poincaré group (Haag-Lopuszanski-Sohnius extension of the Coleman-Mandula theorem). And this extension is supersymmetry, with the properties we were looking for above. More precisely, one may introduce fermionic generators $\mathbf{Q}_\alpha$, in addition to the bosonic symmetry generators ($\mathbf{P}_\mu$ for translations and $\mathbf{M}_{\mu\nu}$ for proper Lorentz transformations), which satisfy the following algebra:

$$\{\mathbf{Q}_\alpha, \overline{\mathbf{Q}}_\beta\} = 2\sigma^\mu_{\alpha\beta}\mathbf{P}_\mu\,, \tag{8.3}$$

$$\{\mathbf{Q}_\alpha, \mathbf{Q}_\beta\} = \{\overline{\mathbf{Q}}_\alpha, \overline{\mathbf{Q}}_\beta\} = 0\,, \tag{8.4}$$

$$[\mathbf{Q}_\alpha, \mathbf{P}_\mu] = 0\,, \tag{8.5}$$

$$[\mathbf{Q}_\alpha, \mathbf{M}_{\mu\nu}] = i\,(\sigma_{\mu\nu})_\alpha^{\ \beta}\, Q_\beta\,. \tag{8.6}$$

The labels α and β are spinor indices taking the values 1 and 2, the bar denotes conjugation and the algebra involves anticommutators and commutators. Another important point to note is that in eqs. (8.3), (8.5) and (8.6) the new generators mix with the other Poincaré generators.

A theory is supersymmetric if it is invariant under the group of transformations generated by $\mathbf{P}_\mu$, $\mathbf{M}_{\mu\nu}$ and $\mathbf{Q}_\alpha$.

In such a theory, for every bosonic state there is a fermionic state with the same energy, and vice-versa. This follows directly from the fact that the Hamiltonian ($\mathbf{P}_0$) commutes with $\mathbf{Q}$.

Another interesting feature is that the cosmological constant vanishes: the Hamiltonian is bounded from below and the ground state has zero energy (if SUSY is not spontaneously broken). To understand this we simply have to note that since σ^0 is equal to the unit matrix and $\mathbf{P}_0$ is the Hamiltonian, eq. (8.3) entails

$$\{\mathbf{Q}_\alpha, \overline{\mathbf{Q}}_\beta\} = 2\delta_{\alpha\beta}\mathbf{H}\,. \tag{8.7}$$

From this we conclude for an arbitrary state $|\psi\rangle$

$$\langle\psi|\mathbf{H}|\psi\rangle = \langle\psi|\mathbf{Q}\overline{\mathbf{Q}}|\psi\rangle = \|\overline{\mathbf{Q}}|\psi\rangle\| \geq 0\,. \tag{8.8}$$

At the same time we see that

$$\langle\psi|\mathbf{H}|\psi\rangle = 0 \quad \Leftrightarrow \quad \overline{\mathbf{Q}}|\psi\rangle = 0\,, \tag{8.9}$$

which is precisely the condition for SUSY not to be spontaneously broken (see eq. (4.18)).

8.3 The Supersymmetric Harmonic Oscillator

In this subsection we will consider the simplest supersymmetric model and convince ourselves that this model indeed has all the nice properties we expect.

Let us start with the usual (bosonic) harmonic oscillator. The Hamiltonian is given by

$$H_B = \frac{1}{2}\left(p^2 + \omega_B^2\, x^2\right)\,. \tag{8.10}$$

If we define creation and annihilation operators

$$a \equiv \frac{1}{\sqrt{2\omega_B}}\,(p - i\omega_B x)\,, \qquad a^+ \equiv \frac{1}{\sqrt{2\omega_B}}\,(p + i\omega_B x)\,, \tag{8.11}$$

then the canonical commutation relation $[p, x] = -i$ entails the usual commutation relations for the creation and annihilation operators

$$[a, a^+] = 1\,, \quad [a, a] \;=\; [a^+, a^+] \;=\; 0\,. \tag{8.12}$$

If we write the Hamiltonian eq. (8.10) in terms of the creation and annihilation operators, we get

$$H_B \;=\; \frac{\omega_B}{2}\left(a^+\, a + a\, a^+\right) \;=\; \omega_B\left(N_B + \frac{1}{2}\right), \tag{8.13}$$

where we have defined the counting operator $N_B \equiv a^+\, a$. The energy spectrum of this Hamiltonian (i.e. its eigenvalues) is given by

$$E_{n_B} = \omega_B\left(n_B + \frac{1}{2}\right) \quad \text{with} \quad n_B = 0, 1, 2, 3, \ldots\,. \tag{8.14}$$

A point to note is that the ground state energy E_0 is 1/2 and not 0. In a quantum field theory this leads to the problem with infinite ground state energy. This problem is solved by normal ordering.

Let us now repeat these steps for a fermionic harmonic oscillator. We introduce fermionic creation and annihilation operators b and b^+. They satisfy

$$\{b, b^+\} \;=\; 1\,, \quad \{b, b\} \;=\; \{b^+, b^+\} \;=\; 0\,. \tag{8.15}$$

These relations correspond to eq. (8.12). However, since we are dealing with fermionic operators now, the commutators are replaced by anticommutators. In analogy to eq. (8.13) we write the Hamiltonian of the fermionic harmonic oscillator as

$$H_F \;=\; \frac{\omega_F}{2}\left(b^+ b - b\, b^+\right) \;=\; \omega_F\left(N_F - \frac{1}{2}\right)\,, \tag{8.16}$$

where we have introduced another counting operator, $N_F \equiv b^+ b$. Note that there is a relative minus sign between the $b^+ b$ and $b\, b^+$ term. This sign is due to the fermionic nature of the creation and annihilation operators.

The energy spectrum of this Hamiltonian is given by

$$E_{n_F} = \omega_F\left(n_F - \frac{1}{2}\right) \quad \text{with} \quad n_F = 0,\, 1\,. \tag{8.17}$$

Note that contrary to eq. (8.14), n_F can only take the values 0 or 1. This is a reflection of Pauli's exclusion principle in that there cannot be two fermions in the same state.

If we wish we can find an explicit representation of the creation and annihilation operators in terms of Pauli matrices,

$$\begin{aligned} b \;&=\; \sigma_1 - i\sigma_2 \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ b^+ \;&=\; \sigma_1 + i\sigma_2 \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{8.18}$$

In this representation the Hamiltonian eq. (8.16) is given by

$$H_F = \frac{\omega_F}{2}\,\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{8.19}$$

and we see that the eigenvalues of H_F are indeed $\pm\omega_F/2$ as given in eq. (8.17).

Now we are ready to combine the fermionic and the bosonic harmonic oscillator. If we just add the two, we do not increase the symmetry of the theory. In order to do this we also have to require $\omega_B = \omega_F \equiv \omega$. Only in this case do we end up with a supersymmetric model. The Hamiltonian then is

$$H \equiv H_B + H_F\Big|_{\omega_B=\omega_F\equiv\omega} = \frac{\omega}{2}\left(a^+ a + a\, a^+ + b^+ b - b\, b^+\right) = \omega\left(a^+ a + b^+ b\right). \quad (8.20)$$

First of all we naively see that H has an additional symmetry $a \leftrightarrow b$. A state is now determined by two quantum numbers n_B and n_F, and the energy spectrum is

$$E_{n_B,n_F} = \omega\,(n_B + n_F) \quad \text{with} \quad n_B = 0,1,2,3,\ldots,\ n_F = 0,1\,. \quad (8.21)$$

Note that the ground state energy is $E_{0,0} = 0$. Thus as advertised above, the ground state has zero energy. This is simply because the bosonic ground state energy $+1/2$ and the fermionic ground state energy $-1/2$ cancel.

The other feature mentioned before, namely that the states appear in pairs (a fermionic and a bosonic state) with the same energy can be seen from eq. (8.21). Indeed, the states $|n_B, n_F = 0\rangle$ and $|n_B - 1, n_F = 1\rangle$ have the same energy. Furthermore $|n_B, n_F = 0\rangle$ is a bosonic state (integer spin), whereas $|n_B - 1, n_F = 1\rangle$ is a fermionic state (half-integer spin).

8.4 Supercharges

In this subsection we want to look at the symmetry found in subsection 8.2 in a somewhat more formal way.

Through the Noether theorem, a symmetry is related to a conserved current and a conserved charge. Thus, in a supersymmetric theory there is a conserved supercurrent and a conserved supercharge. It is the latter that generates the transformations and we denote it by $\mathbf{Q}$. Since it is conserved it has to commute with the Hamiltonian.

For the supersymmetric harmonic oscillator the supercharge is given by

$$\mathbf{Q}_1 = \sqrt{\omega}\left(a^+ b + a\, b^+\right), \qquad \mathbf{Q}_2 = i\sqrt{\omega}\left(a^+ b - a\, b^+\right), \quad (8.22)$$

where, as mentioned after eq. (8.3), $\mathbf{Q}$ has a spinor index. We now show that the supercharges as defined in eq. (8.22) have the desired properties. Using the (anti-) commutation rules for the creation and annihilation operators, eqs. (8.12) and (8.15), we can compute

$$\begin{aligned}\{\mathbf{Q}_1, \mathbf{Q}_1\} &= \omega\{a^+ b + a\, b^+, a^+ b + a\, b^+\} \\ &= \omega\{a^+ b, a\, b^+\} + \omega\{a\, b^+, a^+ b\} \\ &= 2\omega\left(a^+ a\,(1 - b^+ b) + (1 + a^+ a)\, b^+ b\right) \\ &= 2\omega(a^+ a + b^+ b) = 2H\,. \end{aligned} \quad (8.23)$$

In a similar way we can compute the remaining anticommutators of $\mathbf{Q}_1$ and $\mathbf{Q}_2$, and we get

$$\{\mathbf{Q}_1, \mathbf{Q}_1\} \;=\; \{\mathbf{Q}_2, \mathbf{Q}_2\} \;=\; 2\mathbf{H}\,, \qquad \{\mathbf{Q}_1, \mathbf{Q}_2\} \;=\; 0\,. \tag{8.24}$$

Note that this is in agreement with eq. (8.7). Now we can see that, as promised, the supercharge is conserved:

$$[\mathbf{Q}_1, \mathbf{H}] \;=\; \left[\mathbf{Q}_1, (\mathbf{Q}_1)^2\right] \;=\; 0\,, \tag{8.25}$$

$$[\mathbf{Q}_2, \mathbf{H}] \;=\; \left[\mathbf{Q}_2, (\mathbf{Q}_2)^2\right] \;=\; 0\,. \tag{8.26}$$

Eqs. (8.25) and (8.26) allow us to see that the states in this theory come in pairs. In fact, let $|\Psi\rangle$ be an eigenstate of $\mathbf{H}$, i.e. $\mathbf{H}|\Psi\rangle = E_\Psi|\Psi\rangle$. Then $\mathbf{Q}_1|\Psi\rangle$ is an eigenstate of $\mathbf{H}$ with the same energy,

$$\mathbf{H}\mathbf{Q}_1|\Psi\rangle \;=\; \mathbf{Q}_1\mathbf{H}|\Psi\rangle \;=\; \mathbf{Q}_1 E_\Psi|\Psi\rangle \;=\; E_\Psi\mathbf{Q}_1|\Psi\rangle\,. \tag{8.27}$$

If $|\Psi\rangle$ is a bosonic state containing n_B bosons and no fermions, then

$$\mathbf{Q}_1|\Psi\rangle \;=\; \mathbf{Q}_1|n_B, 0\rangle \;=\; \sqrt{\omega}\left(a^+\, b + a\, b^+\right)|n_B, 0\rangle \;=\; |n_B - 1, 1\rangle \tag{8.28}$$

is a fermionic state with the same energy. Similarly, if $|\Psi\rangle = |n_B, 1\rangle$ is a fermionic state, then $\mathbf{Q}_1|\Psi\rangle = |n_B + 1, 0\rangle$ is a bosonic state with the same energy. Thus the states come indeed in pairs with the same energy, one fermionic and one bosonic.

Of course, the same argument could have been made with $\mathbf{Q}_2$ rather than with $\mathbf{Q}_1$. However, $\mathbf{Q}_2$ acting on a state $|\Psi\rangle$ produces the same state as $\mathbf{Q}_1$ acting on a state $|\Psi\rangle$. Thus, there are not four but only two states with the same energy.

What we have seen is that if we start with the usual bosonic harmonic oscillator and want to make this theory supersymmetric, then we are led to introduce for every bosonic (fermionic) state a fermionic (bosonic) state with the same energy. This is exactly what happens if we want to make the Standard Model supersymmetric: For each boson (fermion) we have to introduce a fermionic (bosonic) partner, thereby doubling the particle spectrum.

8.5 Superfields

The superfield is a very convenient piece of SUSY notation, which rests on the abstract idea of supersymmetrising space-time. Suppose that for the four (bosonic) dimensions we know, that is x, y, z and t, we add a pair of fermionic dimensions η and $\bar{\eta}$. The SUSY transformations Q and $\bar{Q}$ are translations in the fermionic directions of this "superspace". Being η and $\bar{\eta}$ fermions, they anticommute with themselves, so the Taylor expansion in these fermionic dimensions ends quickly!

The superfield associated with, say, the Higgs, is a function of superspace:

$$H(x^\mu, \eta) \;=\; H(x^\mu) + \eta\, h(x^\mu) + \eta\,\eta\, F(x^\mu)\,. \tag{8.29}$$

H is an example of a (left-handed) "chiral superfield", a simple sort of superfield that is independent of $\bar{\eta}$, suitable for decribing a matter multiplet made of a left-handed fermion and complex scalar. By a standard abuse of notation, the superfield has the same symbol as its scalar component. So on the RHS of the equality, H is the scalar Higgs, h is the higgsino, and F is a bosonic field of mass dimension two, which therefore cannot have kinetic terms and can be removed from the Lagrangian by using its equations of motion (something like a Lagrange multiplier). We make no more mention of F, other than to note that it is the origin of calling part of the SUSY Lagrangian "F-terms".

The reason that superfields are convenient, is that one can compactly write all the SM Yukawa interactions, and their supersymmetric relatives (of which there are very many), as the "superpotential":

$$W \;=\; Y_e H_d L E^c + Y_\nu H_u L N^c + Y_d H_d L D^c + Y_u H_u L U^c\,. \tag{8.30}$$

For simplicity, let us consider only one generation. Y_f is the Yukawa coupling for fermion f, and the right-handed fermions (*e.g.* $\overline{e_R}$) have been written as left-handed anti-particles (e^c). Notice that there are two physically distinct Higgs doublets H_u and H_d, where in the SM we have used one doublet and its charge conjugate. We will return later to the reason for this extra field.

To obtain supersymmetric interactions of component fields, in ordinary four-dimensional space, one should extract the F-term of W. That is, expand each field as in eq. (8.29) and pick out all the terms $\propto \eta^2$. It is clear that this will include the SM Yukawa couplings, because each fermion comes with an η. It also gives scalar four point interactions. The full expression is

$$\begin{aligned} \mathcal{L}_{SSM} \;&=\; \text{kinetic terms} + \sum_{ij} \frac{\partial^2 W}{\partial \Phi_i \partial \Phi_j} \psi_i \psi_j - \sum_k |\frac{\partial W}{\partial \Phi_i}|^2 \\ &=\; \ldots + Y_e H_d \ell e^c + \ldots - |Y_e|^2 (H_d L)(H_d L)^* - \ldots\,, \end{aligned} \tag{8.31}$$

where i, j, k run over all the superfields in W, and on the second line are the parts coming from derivatives with respect to E^c. The fermion index contraction is in the same shorthand as eq. (7.30). The kinetic terms and gauge interactions come from another function of the superfields.

It is possible to draw diagrams and do calculations in superspace; this can be useful for obtaining exact supersymmetric cancellations.

8.6 The MSSM Particle Content (Partially)

The Lagrangian for the Minimal Supersymmetric SM (MSSM) can be motivated as follows:

1. Add a boson for all SM fermions, and a fermion for all SM bosons.

2. "Supersymmetrise" the SM Feynman diagrams.

3. Observe that step 2 gave superpartners with the same masses as their SM relatives. As we have not observed any superpartners, add "SUSY breaking" mass terms to make them heavier than current experimental sensitivities. (These masses are called "soft" because the quadratic divergences still cancel — as you have discovered in the problem.)

This heuristic recipe will give a Lagrangian with ~ 125 free parameters, compared to 19 in the SM. The vast majority of the additional parameters come in the SUSY breaking sector and make the theory unwieldy to study. It is therefore common to work within simplified SUSY breaking scenarios with fewer parameters, like e.g. mSUGRA, the minimal version of supergravity grand unification. In this model universality of the soft SUSY breaking parameters is assumed (there are only four additional new parameters plus one sign), leading to a suppression of flavour changing neutral currents.

In this subsection we restrict outselves to the first step outlined above, describing the particle content of the MSSM. Feynman rules can be found elsewhere.

Superpartners are often written as capitalised, or "tilded" SM particles. The partners of one generation of SM leptons are a slepton doublet, a singlet selectron and a "right-handed" sneutrino:

$$\begin{array}{llll} \ell = \begin{pmatrix} e_L \\ \nu_L \end{pmatrix} & \to \ \tilde{\ell} = \begin{pmatrix} \tilde{e}_L \\ \tilde{\nu}_L \end{pmatrix} & \text{or} \ \ L = \begin{pmatrix} E_L \\ N_L \end{pmatrix}, \\ e^c & \to \ \widetilde{e^c} & \text{or} \ \ E^c\,, \\ (\nu_R)^c & \to \ \widetilde{(\nu_R^c)} & \text{or} \ \ N^c\,, \end{array} \tag{8.32}$$

and sometimes, abusively, the ^{c} is dropped from the singlets, although they remain "left-handed". Similarly, one introduces squark partners, of all colours and flavours, for the quarks.

The spartners of the SM bosons are the "-inos", who can be names according to whether they are added before (Bino and three Winos) or after (Photino, Zino and two Winos) spontaneous symmetry breaking:

$$\begin{array}{lll} \gamma & \rightarrow & \tilde{\gamma}\,, \\ Z & \rightarrow & \tilde{z} \ \text{ or } \ z\,, \\ W^{\pm} & \rightarrow & \tilde{w}^{\pm} \ \text{ or } \ w^{\pm}\,, \\ H = \begin{pmatrix} H^{+} \\ H_0 \end{pmatrix} & \rightarrow & \tilde{h}_u = \begin{pmatrix} \tilde{h}_u^{+} \\ \tilde{h}_u^{0} \end{pmatrix}. \end{array} \tag{8.33}$$

In supersymmetry, we need a second Higgs doublet. One can see this from the formal structure of the theory, or from considerations of anomaly cancellation, or by counting fermionic degrees of freedom. Let us do the last: Suppose we break the electroweak gauge symmetry in an exactly supersymmetric SM. The spartners must therefore have the same masses as the SM particles, and notice in the SM after spontaneous symmetry breaking, there are no massless charged bosons. However, among the inos in eq. (8.33), there are three *chiral* charged fermions, and it takes two chiral fermions to make a massive charged "Dirac" fermion (a Majorana mass would break charge conservation). The solution to this problem is to add a second Higgs,

$$H_d = \begin{pmatrix} H^{0} \\ H^{-} \end{pmatrix} \rightarrow \tilde{h}_d = \begin{pmatrix} \tilde{h}_d^{0} \\ \tilde{h}_d^{-} \end{pmatrix} \tag{8.34}$$

which gives mass to the d quarks and charged leptons.

Recall that we must add soft masses for all these new fermions, to ensure that they should not have been discovered yet, so the physical mass eigenstates will be four neutralinos and two (four component fermion) charginos, respectively linear combinations of $\tilde{\gamma}, \tilde{z}, h_u$ and h_d, and $\tilde{w}^{\pm}, h_u^{+}, h_d^{-}$.

8.7 Summary

- Supersymmetry transforms bosons $\leftrightarrow$ fermions. It is an (the only possible) extension of the Poincaré algebra.
- Since fermion loops come with a relative minus sign, the Higgs mass would have no quadratic divergence in an exactly supersymmetric theory.
- To supersymmetrise the SM, one has to add a boson (sfermion) for every fermion, and a fermion (-ino) for every boson. Then one adds a second Higgs doublet and its SUSY partners.
- No spartners have been observed so far, so one gives them masses in excess of current experimental bounds. This breaks the supersymmetry, and allows finite corrections to the Higgs mass.

- At the time of writing this sentence, supersymmetry has not been found.

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